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Discrete collapsing sandpile model☆

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ABSTRACT

Our main goal is to introduce and study a discrete model for the collapsing of a pile of cubes. This is a typical example of self-organized critical phenomena exhibited by a critical slope. We prove the existence and uniqueness of the solution for the model. Then by using dual arguments we study the numerical computation of the solution and we present some numerical simulations.

1. Introduction

The dynamics of granular materials have been studied quite intensively due to their importance in various naturally occurring phenomena such as landslides, rockfalls, desert dunes evolution, sediment transport in rivers, . . . and engineering transportation applications. The description of such flows still represents a major challenge for the theory. In the last decade, several studies have been devoted to the mathematical and numerical studies of a granular system. Different models have been proposed using the kinetic approach (cf. [1,2]), cellular automata (cf. [3–6]) or partial differential equations (cf. [7–20]).

Granular materials are complex objects and it is important to understand their behavior by using simple prototypes. Actually, it is known that one of the approach that may be relevant for their study is based on modeling the dynamic of a pile of cubes. That is, to imagine that the matter at the microscopic level consists of particles similar to cubes (in some cases, a particle can be likened to a certain volume of a material) arranged on a regular grid. The principle after consists in establishing simple rules across the unit cell and repeat until the interplay between cells occurs by itself coherent structures or organized forms at the macroscopic scale. Of course, the elementary constituents of a material are so numerous that the study at the microscopic level needs probabilistic methods. However, appropriate scaling of time enables a transition to deterministic models of nonlocal type (see for instance [5,21]). These rescaling takes into account rigorously the fact that there is a very large number of particles and there is a significant gap between the time scales of microscopic and macroscopic.

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A typical example is the growing pile of cubes (cf. [5]) which corresponds to the evolution of a stack of unit cubes resting on the plane when new cubes are being added to the pile. In [5], Evans and Rezakhanlou introduce a stochastic description of the dynamics and proved that, if we randomly add more and more, smaller and smaller cubes, we obtain an interesting continuum limit, which is an evolution governed by the sub differential of a convex functional that is very connected to the Prigozhin model for sandpile [10]. To that aim, they introduce an intermediate nonlinear discrete dynamic of nonlocal type at the level of cubes. By using the Partial Integro-Differential Equation, Igbida shows in [21] that this discrete nonlocal equation gives a right deterministic description of the dynamic of a growing pile of granular structures when the component is not very small. Our aim here is to show how to use this kind of discrete equation to model the collapsing of an unstable pile of cubes.

The paper is organized as follows: in the next section we establish our discrete model and study the existence and uniqueness of the solution. In Section 3 we develop a numerical study of the model based on the duality argument. At last, we give numerical simulations showing the stabilization of unstable discrete structures.

2. The discrete model for the collapse of a pile

It is well known by now, that the collapsing phenomena in granular materials can be described by nonlinear evolution equations governed by nondecreasing critical angles. In the continuous case, recall that combining the continuity equation of fluid dynamic and the phenomenological equation, N. Igbida introduce in [20] (see also [16,17]) a sub-gradient flow for variational problems with time dependent gradient constraints. The gradient constraints are interpreted as the critical angle of a sandpile. In particular, the continuous model [20] produces an evolution in time of avalanches in a drying of a sandpile, rather than instantaneous collapse. Our aim here is to introduce a discrete non local model that we can associate with such phenomena.

2.1. The discrete model

We consider the surface of the pile be divided into cubes of integer point $i \in \mathbb{Z}^n$, $n = 1$ or 2. So, a stack of cubes can be described by an application $u : \mathbb{Z}^n \to \mathbb{R}$, where $u(i)$ describes the density of cubes at the position i.

The collapse is produced when the slope of the surface exceeds an angle of stability. In the discrete case the stability condition for a profile u reads (cf. [5,21])

$$|u(i) - u(j)| \leq 1 \quad \text{for } i \sim j, \tag{1}$$

where we use $i \sim j$ to describe $|i - j| \leq 1$. Assume that, we start with unstable configuration represented by $u_0 : \mathbb{Z}^n \to \mathbb{R}$ such that

$$|u_0(i) - u_0(j)| > 1 \quad \text{for } i \sim j.$$

To reach a stable configuration, we assume a suitable of various avalanches is produced, so as to stabilize the pile. More precisely, we assume that the pile tends to stabilize itself by taking a continuous sequence of intermediate profiles characterized by

$$|u(i) - u(j)| \leq c(t) \quad \text{for } i \sim j, \tag{2}$$

where $c : [0, T) \to \mathbb{R}^+$ is a given non increasing function satisfying

$$\lim_{t \to T} c(t) = 1.$$

Here, the stability constraint, forces the pile to rearrange itself to reach a stable profile. So, a suitable of various unstable configurations is produced with non increasing angle of stability that converges to 1, as $t \to T \leq \infty$.

The dynamic of the height $u(t, i)$ of the pile at a fixed point $i \in \mathbb{Z}^n$, can be derived as follows. For a small time Δt, the evolution of u is given by:

$$u(t + \Delta t, i) \simeq u(t, i) + \Delta t Q(t, i),$$

where $Q(t, i)$ is the rate of the material arriving at the position i. We can express Q as follows

$$Q(t, i) = I(t, i) - O(t, i),$$

where, $I(t, i)$ records the material arriving to the position i from the neighborhood positions and $O(t, i)$ records the material leaving the position i towards neighborhood positions. We have

$$I(t, i) = \sum_{j : j \sim i} \alpha(t, j, i) \quad \text{and} \quad O(t, i) = \sum_{j : j \sim i} \alpha(t, i, j),$$

where $\alpha(t, i, j)$ records the material arriving to the position j from the neighborhood positions i. This implies that

$$\frac{u(t + \Delta t, i) - u(t, i)}{\Delta t} + \sum_{j : j \sim i} (\alpha(t, i, j) - \alpha(t, j, i)) \simeq 0.$$