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Sobolev type embeddings and an inhomogeneous quasilinear elliptic equation on $\mathbb{R}^N$ with singular weights☆

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ABSTRACT

In this paper we build a Sobolev type embedding from a weighted Sobolev space into a weighted L^1 space on the entire space $\mathbb{R}^N$. As applications, we study the existence and the uniqueness of radial solutions of the inhomogeneous quasilinear elliptic equation

$$\begin{cases} -\mathrm{div}(|\nabla u|^{p-2}\nabla u) + V(|x|)|u|^{q-2}u = Q(|x|), & x \in \mathbb{R}^N, \\ u(x) \to 0, \quad |x| \to \infty \end{cases}$$

with singular radial weighted functions V and Q.

1. Introduction

In this paper we study the quasilinear elliptic equation

$$\begin{cases} -\mathrm{div}(|\nabla u|^{p-2}\nabla u) + V(|x|)|u|^{q-2}u = Q(|x|), & x \in \mathbb{R}^N, \\ u(x) \to 0, \quad |x| \to \infty \end{cases} \tag{P}$$

where $N \geqslant 3$, $1 < p, q < N$ and $p \neq q$ which means that the left hand side of (P) is inhomogeneous in u. The functions $V, Q \in C((0,\infty), (0,\infty))$ satisfy the following properties near zero and infinity:

(V) there exist $a, a_0 \in \mathbb{R}$ such that

$$\liminf_{r\to\infty} \frac{V(r)}{r^a} > 0, \qquad \liminf_{r\to 0} \frac{V(r)}{r^{a_0}} > 0,$$

(Q) there exist $b, b_0 \in \mathbb{R}$ such that

$$\limsup_{r\to\infty} \frac{Q(r)}{r^b} < \infty, \qquad \limsup_{r\to 0} \frac{Q(r)}{r^{b_0}} < \infty,$$

From (V) and (Q) one sees that the potentials V and Q may be singular in the sense that they are unbounded, decaying or vanishing.

The purpose of this paper is to prove the existence and the uniqueness of a solution of (P) under certain circumstance by variational methods.

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We introduce some notations of function spaces. Let $C_0^\infty(\mathbb{R}^N)$ be the set consisting of all the smooth functions with compact support and $C_{0,r}^\infty(\mathbb{R}^N) = \{u \in C_0^\infty(\mathbb{R}^N) \mid u \text{ is radial}\}$. Let $D_r^{1,p}(\mathbb{R}^N)$ be the completion of $C_{0,r}^\infty(\mathbb{R}^N)$ under

$$\|\nabla u\|_{L^p(\mathbb{R}^N)} := \left(\int_{\mathbb{R}^N} |\nabla u|^p dx\right)^{\frac{1}{p}}.$$

Define

$$L^q(\mathbb{R}^N; V) := \left\{u : \mathbb{R}^N \to \mathbb{R} \mid u \text{ is Lebesgue measurable, } \int_{\mathbb{R}^N} V(|x|)|u|^q \mathrm{d}x < \infty\right\},$$

and

$$L^1(\mathbb{R}^N; Q) := \left\{u : \mathbb{R}^N \to \mathbb{R} \mid u \text{ is Lebesgue measurable, } \int_{\mathbb{R}^N} Q(|x|)|u| dx < \infty\right\},$$

with norms

$$\|u\|_{L^q(\mathbb{R}^N;V)} := \left(\int_{\mathbb{R}^N} V(|x|)|u|^q \mathrm{d}x\right)^{\frac{1}{q}},$$

and

$$\|u\|_{L^1(\mathbb{R}^N;Q)} := \int_{\mathbb{R}^N} Q(|x|)|u| \mathrm{d}x.$$

Then we define

$$X_r(\mathbb{R}^N; V) := D_r^{1,p}(\mathbb{R}^N) \cap L^q(\mathbb{R}^N; V),$$

which is a reflexive Banach space equipped with the norm (see [1–3])

$$\|u\|_{X_r(\mathbb{R}^N;V)} := \|\nabla u\|_{L^p(\mathbb{R}^N)} + \|u\|_{L^q(\mathbb{R}^N;V)}.$$

If there is a continuous embedding from $X_r(\mathbb{R}^N; V)$ into $L^1(\mathbb{R}^N; Q)$, then the functional

$$\Phi(u) := \frac{1}{p}\int_{\mathbb{R}^N} |\nabla u|^p dx + \frac{1}{q}\int_{\mathbb{R}^N} V(|x|)|u|^q dx - \int_{\mathbb{R}^N} Q(|x|) u dx$$

is well defined and is of C^1 on $X_r(\mathbb{R}^N; V)$. A critical point $u \in X_r(\mathbb{R}^N; V)$ of Φ is exactly a weak radial solution of (P) in the sense that $u \in X_r(\mathbb{R}^N; V)$ satisfies

$$\int_{\mathbb{R}^N} |\nabla u|^{p-2}\nabla u \nabla \varphi + V(|x|)|u|^{q-2}u\varphi \, dx = \int_{\mathbb{R}^N} Q(|x|)\varphi \, dx$$

for all $\varphi \in X_r(\mathbb{R}^N; V)$.

In Section 2 we build a compact embedding from $X_r(\mathbb{R}^N; V)$ into $L^1(\mathbb{R}^N; Q)$ in the basis of the assumptions on V and Q. Comments, remarks and comparisons are included in this section. In Section 3 we prove the existence of a unique weak solution of (P).

2. The embedding $X_r(\mathbb{R}^N; V) \hookrightarrow L^1(\mathbb{R}^N; Q)$

In this section we build an embedding theorem from $X_r(\mathbb{R}^N; V)$ into $L^1(\mathbb{R}^N; Q)$ which is continuous and compact. Denote by B_ρ the ball in $\mathbb{R}^N$ centered at 0 with radius $\rho > 0$. For $A \subset \mathbb{R}^N$, A^c denotes the complement of A in $\mathbb{R}^N$.

We cite several radial lemmas established in [4,5].

Lemma 2.1 ([4]). *Let $1 < p < N$. Then there exists $\widehat{C} = \widehat{C}(N, p) > 0$, such that for all $u \in D_r^{1,p}(\mathbb{R}^N)$,*

$$|u(x)| \leqslant \widehat{C}|x|^{-\frac{N-p}{p}} \|\nabla u\|_{L^p(\mathbb{R}^N)}.$$

Lemma 2.2 ([5]).

(i) *Let $1 < q < p < N$. Assume* (V) *with $a > \frac{q(p-N)}{p(q-1)} - N$. Then for*

$$0 < \alpha < \alpha^* := \frac{N-p}{p} + \frac{q-1}{q}(a+N),$$

there exist $R \geqslant 1$ and $\widetilde{C} = \widetilde{C}(N, p, q, a, \alpha) > 0$ such that for all $u \in X_r(\mathbb{R}^N; V)$,

$$|u(x)| \leqslant \widetilde{C}|x|^{-\frac{\alpha}{q}} \|u\|_{X_r(\mathbb{R}^N;V)}, \quad |x| \geqslant R.$$