Nonlinear Analysis 96 (2014) 87–95

Contents lists available at ScienceDirect

Nonlinear Analysis

journal homepage: www.elsevier.com/locate/na

Imbedding theorems in Orlicz–Sobolev space of differential forms

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ARTICLE INFO

Article history:
Received 13 April 2013
Accepted 7 November 2013
Communicated by S. Carl

MSC:
primary 35J60
secondary 35B45
30C65
47J05
46E35

Keywords:
Imbedding inequalities
Differential forms
Orlicz norms
Harmonic equations

ABSTRACT

In this paper, we prove imbedding inequalities with L^{φ}-norms in the Orlicz–Sobolev space of the forms and establish L^{φ} norm inequalities for the related operators applied to differential forms. We also obtain the global imbedding theorems in $L^{\varphi}(m)$-averaging domains.

1. Introduction

The purpose of this paper is to develop the local and global L^{φ} imbedding inequalities for the Orlicz–Sobolev space of differential forms satisfying the A-harmonic equation. The imbedding inequalities have been playing a crucial role in the L^p theory of the Sobolev space and partial differential equations. The study and applications of imbedding inequalities are now ubiquitous in different areas, including PDEs and analysis. The investigation of the A-harmonic equation for differential forms has developed rapidly in recent years. The A-harmonic equation is an important extension of the p-harmonic equation $\mathrm{div}(\nabla u|\nabla u|^{p-2}) = 0$ in $\mathbb{R}^n$, $p > 1$. In the meantime, the p-harmonic equation is a natural generalization of the usual Laplace equation $\Delta u = 0$. Many interesting results concerning the properties of solutions to the A-harmonic equation have been established recently, see [1–5]. As extensions of the functions, differential forms have been widely studied and used in many fields of sciences and engineering, including theoretical physics, general relativity, potential theory and electromagnetism. For instance, differential forms can be used to describe various systems of partial differential equations and to express different geometrical structures on manifolds. Some of them are often utilized in studying deformations of elastic bodies, the related extrema for variational integrals and certain geometric invariance. The norm estimates for functions or differential forms are critical to investigate the properties of the solutions of the partial differential equations, or a system of the partial differential equations. The study of L^p norm inequalities, including L^p imbedding inequalities, for differential forms satisfying some versions of harmonic equations has been well developed during the recent years, see [1–9]. However, the investigation

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0362-546X/$ – see front matter
http://dx.doi.org/10.1016/j.na.2013.11.005

for L^{φ} imbeddings in the Orlicz–Sobolev space of differential forms just started. In this paper, we prove the local and global L^{φ} imbedding theorems for the Orlicz–Sobolev space of differential forms. Our main results are presented and proved in Theorems 2.5 and 3.2, respectively. These results enrich the L^p theory of differential forms and can be used to estimate the integrals of the solutions of the related differential system and to study the L^{φ} integrability of differential forms.

Throughout this paper, we always assume that Ω is a bounded domain in $\mathbb{R}^n$, $n \geq 2$, B and σB are the balls with the same center and $\mathrm{diam}(\sigma B) = \sigma \mathrm{diam}(B)$. We use $|E|$ to denote the n-dimensional Lebesgue measure of a set $E \subseteq \mathbb{R}^n$. For a function u, the average of u over B is defined by $u_B = \frac{1}{|B|}\int_B u dm$. All integrals involved in this paper are the Lebesgue integrals. Differential forms are widely used not only in analysis and partial differential equations [1,10], but also in physics [11,12]. Differential forms are extensions of differentiable functions in $\mathbb{R}^n$. For example, the function $u(x_1, x_2, \ldots, x_n)$ is called a 0-form. A differential 1-form $u(x)$ in $\mathbb{R}^n$ can be written as $u(x) = \sum_{i=1}^n u_i(x_1, x_2, \ldots, x_n)dx_i$, where the coefficient functions $u_i(x_1, x_2, \ldots, x_n)$, $i = 1, 2, \ldots, n$, are differentiable. Similarly, a differential k-form $u(x)$ can be expressed as

$$u(x) = \sum_I u_I(x)dx_I = \sum u_{i_1 i_2 \cdots i_k}(x)dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k},$$

where $I = (i_1, i_2, \ldots, i_k)$, $1 \leq i_1 < i_2 < \cdots < i_k \leq n$. Let $\wedge^l = \wedge^l(\mathbb{R}^n)$ be the set of all l-forms in $\mathbb{R}^n$, $D'(\Omega, \wedge^l)$ be the space of all differential l-forms in Ω, and $L^p(\Omega, \wedge^l)$ be the l-forms $u(x) = \sum_I u_I(x)dx_I$ in Ω satisfying $\int_\Omega |u_I|^p < \infty$ for all ordered l-tuples I, $l = 1, 2, \ldots, n$. We denote the exterior derivative by d and the Hodge star operator by $\star$. The Hodge codifferential operator $d^\star$ is given by $d^\star = (-1)^{nl+1} \star d \star$, $l = 1, 2, \ldots, n$. For $u \in D'(\Omega, \wedge^l)$ the vector-valued differential form

$$\nabla u = \left(\frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_n}\right)$$

consists of differential forms $\frac{\partial u}{\partial x_i} \in D'(\Omega, \wedge^l)$, where the partial differentiation is applied to the coefficients of ω. We consider here the nonlinear partial differential equation

$$d^\star A(x, du) = B(x, du) \tag{1.1}$$

which is called non-homogeneous A-harmonic equation, where $A : \Omega \times \wedge^l(\mathbb{R}^n) \to \wedge^l(\mathbb{R}^n)$ and $B : \Omega \times \wedge^l(\mathbb{R}^n) \to \wedge^{l-1}(\mathbb{R}^n)$ satisfy the conditions:

$$|A(x, \xi)| \leq a|\xi|^{p-1}, \qquad A(x, \xi) \cdot \xi \geq |\xi|^p \quad \text{and} \quad |B(x, \xi)| \leq b|\xi|^{p-1} \tag{1.2}$$

for almost every $x \in \Omega$ and all $\xi \in \wedge^l(\mathbb{R}^n)$. Here $a, b > 0$ are constants and $1 < p < \infty$ is a fixed exponent associated with (1.1). A solution to (1.1) is an element of the Sobolev space $W^{1,p}_{loc}(\Omega, \wedge^{l-1})$ such that

$$\int_\Omega A(x, du) \cdot d\varphi + B(x, du) \cdot \varphi = 0 \tag{1.3}$$

for all $\varphi \in W^{1,p}_{loc}(\Omega, \wedge^{l-1})$ with compact support. If u is a function (0-form) in $\mathbb{R}^n$, the Eq. (1.1) reduces to

$$\mathrm{div} A(x, \nabla u) = B(x, \nabla u). \tag{1.4}$$

If the operator $B = 0$, Eq. (1.1) becomes

$$d^\star A(x, du) = 0 \tag{1.5}$$

which is called the (homogeneous) A-harmonic equation. Let $A : \Omega \times \wedge^l(\mathbb{R}^n) \to \wedge^l(\mathbb{R}^n)$ be defined by $A(x, \xi) = \xi|\xi|^{p-2}$ with $p > 1$. Then, A satisfies the required conditions and (1.5) becomes the p-harmonic equation $d^\star(du|du|^{p-2}) = 0$ for differential forms. See [1–9] for recent results on the A-harmonic equations and related topics.

Let $D \subset \mathbb{R}^n$ be a bounded, convex domain. The following operator K_y with the case $y = 0$ was first introduced by H. Cartan in [10]. Then, it was extended to the following general version in [13]. For each $y \in D$, there corresponds a linear operator $K_y : C^\infty(D, \Lambda^l) \to C^\infty(D, \Lambda^{l-1})$ defined by $(K_y\omega)(x; \xi_1, \ldots, \xi_{l-1}) = \int_0^1 t^{l-1}\omega(tx + y - ty; x - y, \xi_1, \ldots, \xi_{l-1})dt$ and the decomposition $\omega = d(K_y\omega) + K_y(d\omega)$. A homotopy operator $T : C^\infty(D, \Lambda^l) \to C^\infty(D, \Lambda^{l-1})$ is defined by averaging K_y over all points y in D

$$T\omega = \int_D \varphi(y)K_y\omega dy\,, \tag{1.6}$$

where $\varphi \in C_0^\infty(D)$ is normalized by $\int_D \varphi(y)dy = 1$. For simplicity purpose, we write $\xi = (\xi_1, \ldots, \xi_{l-1})$. Then, $T\omega(x; \xi) = \int_0^1 t^{l-1}\int_D \varphi(y)\omega(tx + y - ty; x - y, \xi)dydt$. By substituting $z = tx + y - ty$ and $t = s/(1 + s)$, we have

$$T\omega(x; \xi) = \int_D \omega(z, \zeta(z, x - z), \xi)dz, \tag{1.7}$$