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Lorentz estimates for obstacle parabolic problems

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ABSTRACT

We prove that the spatial gradient of (variational) solutions to parabolic obstacle problems of p-Laplacian type enjoys the same regularity of the data and of the derivatives of the obstacle in the scale of Lorentz spaces.

1. Introduction

In this paper we deal with the *obstacle problem* related to the parabolic Cauchy–Dirichlet problem

$$\begin{cases} u_t - \operatorname{div} a(x,t,Du) = f - \operatorname{div}\left[|F|^{p-2}F\right] & \text{in } \Omega_T := \Omega \times (0,T), \\ u = 0 & \text{on } \partial_{\mathrm{lat}}\Omega_T := \partial\Omega \times (0,T), \\ u(\cdot,0) = u_0 & \text{in } \Omega, \end{cases} \tag{1.1}$$

where the vector field models the p-Laplacian operator with coefficients

$$a(x,t,Du) \approx b(x,t)\big(s^2 + |Du|^2\big)^{\frac{p-2}{2}} Du, \quad p > \frac{2n}{n+2}, \; s \in [0,1], \tag{1.2}$$

see (1.8), and where the obstacle ψ is not continuous, as often considered in the literature. We are interested in *sharp integrability estimates* for the gradient Du of solutions to the *variational inequality* related to (1.1) in terms of integrability of the data on the right-hand side f, F and of the obstacle ψ in the *scale of Lorentz spaces*; here $\Omega \subset \mathbb{R}^n$, $n \geq 2$ is a bounded domain and it will be so for the rest of the paper. More precisely, given an obstacle function $\psi : \Omega \times [0,T] \to \mathbb{R}$,

$$\psi \in L^p(0,T;W^{1,p}(\Omega)) \cap C([0,T];L^2(\Omega)) \tag{1.3}$$

such that

$$\partial_t \psi \in L^{p'}(\Omega_T) \quad \text{and} \quad \psi \leq 0 \quad \text{a.e. on } \partial_{\mathrm{lat}}\Omega_T \tag{1.4}$$

and functions

$$F \in L^p(\Omega_T;\mathbb{R}^n) \quad \text{and} \quad f \in L^{p'}(\Omega_T) \tag{1.5}$$

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(with p' we denote the Hölder conjugate of p, i.e., $p' := p/(p-1)$ for $p > 1$), we consider functions $u \in K_0$, where

$$K_0 := \left\{ u \in L^p(0,T; W_0^{1,p}(\Omega)) \cap C([0,T]; L^2(\Omega)) : u \geq \psi \text{ a.e. in } \Omega_T \right\},$$

satisfying the variational inequality

$$\int_0^T \langle \partial_t v, v-u \rangle_{W^{-1,p}\times W_0^{1,p}}\, dt + \int_{\Omega_T} \langle a(x,t,Du), Dv - Du\rangle\, dz$$
$$\geq -\frac{1}{2}\int_\Omega |v(\cdot,0) - u_0|^2\, dx + \int_{\Omega_T} \langle |F|^{p-2}F, Dv - Du\rangle\, dz + \int_{\Omega_T} f(v-u)\, dz \tag{1.6}$$

for any function $v \in K_0'$, with

$$K_0' := \left\{ v \in K_0 : \partial_t v \in L^{p'}(0,T; W^{-1,p'}(\Omega)) \right\};$$

$\langle \cdot,\cdot \rangle_{W^{-1,p}\times W_0^{1,p}}$ denotes the duality pairing crochet between $W_0^{1,p}(\Omega)$ and its dual space $W^{-1,p}(\Omega)$, while $\langle\cdot,\cdot\rangle$ is the scalar product in $\mathbb{R}^n$. We immediately mention that the existence and uniqueness for the problem we are considering can be inferred from [1, Theorem 6.1]. For the initial value we shall assume

$$u_0 \in W_0^{1,p}(\Omega) \quad \text{and} \quad u_0 \geq \psi(\cdot,0) \quad \text{a.e. in } \Omega; \tag{1.7}$$

using an approximation scheme, we can also allow for initial data in $u_0 \in L^2(\Omega)$. The vector fields we treat model the p-Laplacian operator in the following sense: we take $a : \Omega_T \times \mathbb{R}^n \to \mathbb{R}^n$ such that $\partial_\xi a$ is a Carathéodory function and such that the following ellipticity and growth conditions are satisfied:

$$\begin{cases} \langle \partial_\xi a(x,t,\xi)\lambda, \lambda\rangle \geq \nu(s^2+|\xi|^2)^{\frac{p-2}{2}}|\lambda|^2, \\ |a(x,t,\xi)| + |\partial_\xi a(x,t,\xi)|\big(s^2+|\xi|^2\big)^{\frac{1}{2}} \leq L\big(s^2+|\xi|^2\big)^{\frac{p-1}{2}}, \end{cases} \tag{1.8}$$

for almost every $(x,t) \in \Omega_T$ and all $\xi, \xi_1, \xi_2, \lambda \in \mathbb{R}^n$; the structural constants satisfy $0 < \nu \leq 1 \leq L < \infty$, $s \in [0,1]$ is the degeneracy parameter and the exponent p will always satisfy the lower bound $p > \frac{2n}{n+2}$ as in (1.2). Moreover we shall consider the following nonlinear VMO condition in the spirit of [2,3]: defining for balls $B \subset \Omega$ and for all $t \in (0,T)$ and all $\xi \in \mathbb{R}^n$ the averaged vector field

$$(a)_B(t,\xi) := \fint_B a(\cdot,t,\xi)\, dx, \tag{1.9}$$

we require the averaged, normalized modulus of oscillation $\omega_a(R) \in [0,2L]$

$$\omega_a(R) := \sup_{\substack{t\in(0,T),\\ B\in\mathcal{B}_R, \xi\in\mathbb{R}^n}} \left(\fint_B \left(\frac{|a(y,t,\xi) - (a)_B(t,\xi)|}{(s^2+|\xi|^2)^{(p-1)/2}} \right)^2 dy \right)^{\frac{1}{2}} \tag{1.10}$$

where $\mathcal{B}_R$ is the collection of balls $\{B \equiv B_r(x) \subset \Omega : 0 < r \leq R\}$, to satisfy

$$\lim_{R\searrow 0} \omega_a(R) = 0. \tag{1.11}$$

This means that, if we consider the model case in (1.2) with product coefficients $b(x,t) = d(x)h(t)$, we can allow bounded and measurable time-coefficients ($h \in L^\infty(0,T)$) and bounded and VMO spatial ones ($d \in (L^\infty \cap VMO)(\Omega)$); this kind of "nonlinear VMO condition" includes, as particular case, the regularity conditions we assumed in [4,5] for systems. VMO regularity *only with respect to the spatial variables* has been often assumed to prove regularity estimates of this kind, starting from [6,7], in the case without obstacle; see also [8,2].

Finally we are in a position to state the main result of our paper.

Theorem 1.1. *Let $u \in K_0$ satisfy the variational inequality (1.6), where the vector field $a(\cdot)$ satisfies (1.8) and (1.11); moreover suppose that*

$$|D\psi| + |\partial_t\psi|^{1/(p-1)} + |F| + |f|^{1/(p-1)} \in L(\gamma,q) \quad \text{locally in } \Omega_T \tag{1.12}$$

for some $\gamma > p$ and some $q \in (0,\infty]$. Then $|Du| \in L(\gamma,q)$ locally in Ω_T and there exists a radius $R_0 \leq 1$, depending on $n, p, \nu, L, \omega_a(\cdot), \gamma$ and on q in the case $q < \infty$, such that the following local estimate holds, for parabolic cylinders $Q_{2R} \equiv Q_{2R}(z_0) \subset \Omega_T$, with $2R \leq R_0$:

$$|Q_R|^{-\frac{1}{\gamma}} \left\| |Du| + s \right\|_{L(\gamma,q)(Q_R)} \leq c\left(\fint_{Q_{2R}} \big(|Du|+s\big)^p\, dz \right)^{\frac{d}{p}} + c\,|Q_{2R}|^{-\frac{d}{\gamma}} \|\Psi_{2R} + 1\|^d_{L(\gamma,q)(Q_{2R})}, \tag{1.13}$$