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A perturbation result for the Q_γ curvature problem on $\mathbb{S}^n$ ☆

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ABSTRACT

We consider the problem of prescribing the Q_γ curvature on $\mathbb{S}^n$. Using a perturbation method, we obtain existence results for curvatures close to a positive constant.

1. Introduction

Let (M, g_0) be a C^∞ compact Riemannian manifold of dimension $n \geq 3$. The famous Yamabe problem concerns the existence of a metric conformal to g_0 with constant scalar curvature. This corresponds to solve the following partial differential equation

$$-\Delta_{g_0} v + \frac{n-2}{4(n-1)} R_{g_0} v = \frac{n-2}{4(n-1)} R_g v^{\frac{n+2}{n-2}}, \quad v > 0, \tag{1.1}$$

where $R_g \equiv constant$ is the scalar curvature of g.

The linear operator which appears as the first two terms on the left of (1.1) is known as the conformal Laplacian associated to the metric g_0 and denoted as $P_1^{g_0}$. It is conformally covariant in the sense that if f is any smooth function and $g = v^{\frac{4}{n-2}} g_0$ for some $v > 0$, then

$$P_1^{g_0}(vf) = v^{\frac{n+2}{n-2}} P_1^g(f). \tag{1.2}$$

Setting $f \equiv 1$ in (1.2) yields the familiar relationship (1.1) between the scalar curvatures R_{g_0} and R_g. Another conformally covariant operator is

$$P_2^g = (-\Delta_g)^2 - \mathrm{div}_g(a_n R_g g + b_n \mathrm{Ric}_g)d + \frac{n-4}{2} Q_n^g,$$

which was discovered by Paneitz in the 1980s, see [1,2]. Here Q_n^g is the standard Q-curvature, Ric_g is the Ricci curvature of g and a_n, b_n are constants depending on n.

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P_1 and P_2 (in the following, the superscript g is omitted when there is no ambiguity) are the first two of a sequence of conformally covariant elliptic operators, P_k, which exist for all $k \in \mathbb{N}$ if n is odd, but only for $k \in \{1, \ldots, n/2\}$ if n is even. The first construction of these operators was given by Graham–Jenne–Mason–Sparling in [3] (for this reason they are known as the GJMS operators). In that paper, the authors used the ambient metric construction of Fefferman and Graham systematically to construct conformally invariant powers of the Laplacian. In odd dimensions this construction is unobstructed but for dimension $n = 2m$ gives an invariant form of $(-\Delta)^k$ only for $k \le m$. In [4], Graham showed that this result cannot be improved in four dimensions.

This leads naturally to the question whether there exist any conformally covariant pseudodifferential operators of noninteger orders. In [5], the author constructed an intrinsically defined conformally covariant pseudo-differential operator of arbitrary real number order acting on scalar functions. In the work of Graham and Zworski [6], they proved P_k can be realized as residues at the value $\gamma = k$ of a meromorphic family of scattering operators. Using this opinion, a family of elliptic pseudodifferential operators P_γ^g for noninteger γ was given. We will recall this definition in Section 2. An alternative construction of these operators has been obtained by Juhl in [7,8].

In recent years, there are extensive works on the properties of fractional Laplacian operators as non-local operators together with applications to free-boundary value problems and non-local minimal surfaces, for example, [9–13] and so on. Mathematically, $(-\Delta)^\gamma$ is defined as

$$(-\Delta)^\gamma u = C(n,\gamma)\text{P.V.}\int_{\mathbb{R}^n}\frac{u(x)-u(y)}{|x-y|^{n+2\gamma}}dy = C(n,\gamma)\lim_{\varepsilon\to 0^+}\int_{B_\varepsilon^c(x)}\frac{u(x)-u(y)}{|x-y|^{n+2\gamma}}dy.$$

Here P.V. is a commonly used abbreviation for 'in the principal value sense' and $C(n,\gamma) = \pi^{-(2\gamma+n/2)}\frac{\Gamma(n/2+\gamma)}{\Gamma(-\gamma)}$. It is well known that $(-\Delta)^\gamma$ on $\mathbb{R}^n$ with $\gamma \in (0,1)$ is a nonlocal operator. In the remarkable work of Caffarelli and Silvestre [9], the authors express this nonlocal operator as a generalized Dirichlet–Neumann map for a certain elliptic boundary value problem with local differential operators defined on the upper half-space $\mathbb{R}_+^{n+1} = \{(x,t) : x \in \mathbb{R}^n, t > 0\}$. That is, given a solution $u = u(x)$ of $(-\Delta)^\gamma u = f$ in $\mathbb{R}^n$, one can equivalently consider the dimensionally extended problem for $u = u(x,t)$, which solves

$$\begin{cases}\operatorname{div}(t^{1-2\gamma}\nabla u) = 0, & \text{in } \mathbb{R}_+^{n+1},\\ -d_\gamma t^{1-2\gamma}\partial_t u|_{t\to 0} = f, & \text{on } \partial\mathbb{R}_+^{n+1}.\end{cases}$$

Here the positive constant $d_\gamma > 0$ is explicitly given by

$$d_\gamma = 2^{2\gamma-1}\frac{\Gamma(\gamma)}{\Gamma(1-\gamma)}.$$

In the work of Chang and Gonzalez [14], they extended the work of [9] and characterized P_γ as such a Dirichlet-to-Neumann operator on a conformally compact Einstein manifold.

We focus only on the operators P_γ when $\gamma \in (0, \frac{n}{4})$. These operators have the following conformally covariant property: if $g = v^{\frac{4}{n-2\gamma}} g_0$, then

$$P_\gamma^{g_0}(vf) = v^{\frac{n+2\gamma}{n-2\gamma}} P_\gamma^g(f) \tag{1.3}$$

for any smooth function f, see [14]. Generalizing the formula for scalar curvature and the Paneitz–Branson Q-curvature, the Q-curvature for g of order γ, is defined as

$$Q_\gamma^g = P_\gamma^g(1).$$

Thus one can generalize the classical Yamabe problem and consider the 'fractional Yamabe problem': given a metric g_0 on a compact manifold M, find a function $v > 0$ on M such that if $g = v^{\frac{4}{n-2\gamma}} g_0$, then Q_γ^g is constant. By (1.3), one has to solve the equation

$$P_\gamma^{g_0} v = Q_\gamma^g v^{\frac{n+2\gamma}{n-2\gamma}}, \quad v > 0 \text{ on } M$$

with $Q_\gamma^g \equiv constant$. In this direction, we refer the interested readers to the papers [15–19] and the references therein.

Also, one has the 'prescribing Q_γ curvature problem': given a metric g_0 on a compact manifold M and a smooth function Q on M, find $v > 0$ so that if $g = v^{\frac{4}{n-2\gamma}} g_0$, then $Q_\gamma^g = Q$. By (1.3), this amounts to solve

$$P_\gamma^{g_0} v = Q v^{\frac{n+2\gamma}{n-2\gamma}}, \quad v > 0 \text{ on } M.$$

In this paper, we consider the prescribing Q_γ curvature problem on $\mathbb{S}^n$,

$$P_\gamma^{g_c} v = Q v^{\frac{n+2\gamma}{n-2\gamma}}, \quad v > 0 \text{ on } \mathbb{S}^n. \tag{1.4}$$