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An asymptotic property of the Camassa–Holm equation

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ABSTRACT

Asymptotic densities are used to study an asymptotic property of the dispersionless Camassa–Holm equation. An asymptotic density of a global solution is a weak limit of its scaled momentum density along a sequence of time increasing to infinity. For a global solution with non-negative compactly supported initial momentum density, we show that if the asymptotic density is unique, then it is a positive combination of Dirac measures supported in a bounded interval in the non-negative axis with zero as the only possible accumulation point. In other words, if the scaled momentum density does not oscillate as time goes to infinity, the solution behaves as a combination of peakons moving to the right at different speeds. In contrast to many investigations on the topic, our approach is not spectral theoretic and hence is independent of the structure of the isospectral problem associated with the equation.

1. Introduction

We consider the Cauchy problem of the dispersionless Camassa–Holm equation

$$\begin{cases} u_t - u_{xxt} + 3uu_x = 2u_x u_{xx} + uu_{xxx}, & x \in \mathbb{R}, t > 0, \\ u(0, x) = u_0(x) & x \in \mathbb{R} \end{cases} \tag{1.1}$$

where $u = u(t, x)$ is a scalar. Let $h = u - u_{xx}$ be the momentum density. The same problem in terms of h is

$$\begin{cases} h_t + uh_x + 2u_x h = 0, & x \in \mathbb{R}, t > 0, \\ h(0, x) = h_0(x) & x \in \mathbb{R} \end{cases} \tag{1.2}$$

with $h_0 = u_0 - u_{0,xx}$. For $s \geq 3$, $u \in C([0, T]; H^s(\mathbb{R})) \cap C^1([0, T]; H^{s-1}(\mathbb{R}))$ satisfies (1.1) if and only if the corresponding h satisfies (1.2). The Camassa–Holm equation is a model for the unidirectional propagation of water waves in irrotational flow over a flat bed in the shallow water moderate amplitude regime [1–3]. It can also model the propagation of axially symmetric waves in a hyperelastic rod [4]. The local well-posedness for strong solutions are given in [5–7], together with conditions for their global existence and finite time blow-up. For the well-posedness theory for weak solutions, see Bressan and Constantin [8,9]. The equation has received much attention since Camassa and Holm [1] showed that it can model wave breaking phenomena and have peaked solitary wave solutions called peakons, weak solutions of the form $ce^{-|x-ct|}$ with $c > 0$ ('antipeakon' if $c < 0$). They are smooth except at the crest where they are continuous but with different one-sided tangents, similar to the traveling waves of the greatest height which are solutions to the equations for water waves [10–13]. There are multipeakon/antipeakon solutions of the form $\sum_{i=1}^{N} m_i(t)e^{|x-q_i(t)|}$ [1,14], where the $m_i(t)$'s and $q_i(t)$'s

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satisfy a Hamiltonian system. Without causing confusion, we will use the same phrase 'multipeakon/antipeakon' whatever the signs of m_i's are. The shapes of peakons and multipeakon/antipeakons are stable under small perturbations, making them recognizable physically [15–18]. The Camassa–Holm equation is an integrable infinite dimensional Hamiltonian system. It is the compatibility condition for an isospectral problem and a linear evolution equation for the corresponding eigenfunctions, and can be investigated through direct and inverse spectral theory [1,14,19–21].

Numerical computations (see for instance [14,22,23]) indicate that some global solutions evolve into a train of peakons moving at different speeds. Whether this is true in general has been an open problem (see for instance [24]) until the recent work of Eckhardt and Teschl [21]. Previously, Beals, Sattinger and Szmigielski [25] settled the problem for multipeakon/antipeakon solutions by determining the limits of $m_i(t)$'s and $\dot{q}_i(t)$'s (see the expression for such solutions in the last paragraph) and deduced that in large time they tend to superpositions of noninteracting peakons and antipeakons. The problem was also settled for a simplified flow by Loubet [26] and a class of low regularity solutions by Li [27]. The works of El Dika and Molinet on the stability of multipeakon/antipeakons [16,17] can be interpreted as results on this problem. They show that solutions with initial values close to a superposition of peakon and antipeakon profiles will stay close to the superposition of the profiles each of which translated with a (varying) velocity close to its signed height [16, Lemma 4.1] [17, Theorem 2.1]. Recently Eckhardt and Teschl [21] study the direct and inverse spectral theory for the isospectral problem associated with the equation, only assuming the weight to be a finite signed Borel measure. Using the known time evolution of the spectral quantities, they obtain the asymptotic behavior of the solutions and conclude that the peakon train phenomenon appears in general.

In this article, a method involving asymptotic densities used in [28–30] is employed to give another deduction of the phenomenon. In contrast to many of the studies mentioned above, our approach is not spectral theoretic and hence is independent of the structure of the isospectral problem associated with the equation, or even the existence of such a problem. This may be of value in investigating analogous problems for some other equations. To explain our result, we first fix our notations and define asymptotic densities. Let $[a, b] \subset \mathbb{R}$ be a finite interval. Let $\mathcal{M}[a, b]$ be the space of regular Borel measures on $[a, b]$. We will identify such measures with those on $\mathbb{R}$ supported in $[a, b]$. Let $C_c(\mathbb{R})$ be the space of continuous functions with compact support on $\mathbb{R}$. Obviously, the restrictions of $C_c(\mathbb{R})$ functions to a finite interval $[a, b]$ gives $C[a, b]$. For a Banach space X, $k = 0, 1, \ldots, T^* \in (0, \infty]$, we say that v belongs to the set $C^k([0, T^*); X)$ if for all $T < T^*$, v is in the Banach space $C^k([0, T]; X)$.

Definition 1.1. Let $s \geq 3$ and $u \in C([0, \infty); H^s(\mathbb{R}))$ be a global solution of (1.1).

(a) For $t > 0$, $\tilde{h}(t, y) := th(t, ty)$ is called the scaled momentum density of u.

(b) Suppose supp $\tilde{h}(t, \cdot) \subset [a, b]$ for all $t \geq 1$. $\mu \in \mathcal{M}[a, b]$ is called an *asymptotic density* associated with the initial momentum density h_0 if there is a sequence $t_k \to \infty$ as $k \to \infty$ for which

$$\tilde{h}(t_k, \cdot) \rightharpoonup \mu \quad \text{as } t_k \to \infty,$$

where the convergence is the weak-$*$ convergence in $\mathcal{M}[a, b]$, i.e. for all $\psi \in C[a, b]$, $\int_a^b \tilde{h}(t, y)\psi(y)dy \to \langle \mu, \psi \rangle$.

Notice that asymptotic densities associated with h_0 may not be unique.

Suppose that $u_0 \in H^s(\mathbb{R})$ with $s \geq 3$ and $h_0 \geq 0$ is compactly supported. Let u be the global strong solution to (1.1) (see Theorem 2.1). From Lemma 2.4 the scaled momentum density $\tilde{h}(t, \cdot)$ must be supported in some finite interval $[a, b]$ for t sufficiently large. From Lemma 2.3, $\|\tilde{h}(t, \cdot)\|_{L^1(\mathbb{R})}$ is constant in time. Hence the existence of an asymptotic density associated with h_0 is guaranteed. The following is the main result of this article.

Theorem 1.1. *Let $s \geq 3$ and $u \in C([0, \infty); H^s(\mathbb{R})) \cap C^1([0, \infty); H^{s-1}(\mathbb{R}))$ be a global solution to (1.1). Let $h = u - u_{xx}$ be the momentum density. Suppose $h_0(\cdot) = h(0, \cdot) \geq 0$ has compact support. For $t > 0$, let $\tilde{h}(t, y) = th(t, ty)$. Suppose that there is a unique asymptotic density μ associated with h_0, that is, $\tilde{h}(t, \cdot) \rightharpoonup \mu$ as $t \to \infty$. Then there exist finitely or countably infinitely many $m_i, \alpha_i \in [0, \infty)$ such that*

$$\mu = \sum_i m_i \delta_{\alpha_i}, \tag{1.3}$$

where δ_{α_i} is the delta function supported at α_i, and

(a) *$\alpha_i \neq \alpha_j$ if $i \neq j$, and in case there are infinitely many i's, $\alpha_i \to 0$ as $i \to \infty$;*

(b) *$m_i > 0$ and $\sum_i m_i = \|h_0\|_{L^1(\mathbb{R})}$;*

(c) *for any i, $\alpha_i \in [0, M]$, where $M = \|u\|_{L^\infty(\mathbb{R}^+ \times \mathbb{R})}$;*

(d) *for all i, $\alpha_i \leq (3/4)m_i$.*

Notice that $\mu = \sum_i m_i \delta_{\alpha_i}$ is the asymptotic density of a positive combination of non-interacting peakons $v(t, x) = \sum_i (m_i/2)e^{-|x-\alpha_i t|}$, which is not necessarily a solution though we still use the phrase 'asymptotic density' for convenience. Indeed it is straightforward to check that in the distribution sense, $g(t, x) := (v - v_{xx})(t, x) = \sum_i m_i \delta_{\alpha_i t}(x)$. Hence