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Uniform estimates and convexity in capillary surfaces[☆]

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ABSTRACT

We prove uniform convexity of solutions to the capillarity boundary value problem for fixed boundary angle in $(0, \pi/2)$ and strictly positive capillarity constant provided that the base domain $\Omega \subset \mathbb{R}^2$ is sufficiently close to a disk in a suitable C^4-sense.

1. Introduction

The goal of the present note is to investigate convexity properties of classical solutions $u : \overline{\Omega} \to \mathbb{R}$ to the capillarity equation

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = \kappa u, \quad \text{in } \Omega, \tag{1}$$

together with the (fully) nonlinear boundary condition

$$\frac{\nabla u}{\sqrt{1+|\nabla u|^2}} \cdot \nu = \cos(\gamma), \quad \text{on } \partial\Omega. \tag{2}$$

Here, $\Omega \subset \mathbb{R}^2$ denotes a bounded smooth domain with exterior unit normal $\nu : \partial\Omega \to \mathbb{S}^1$, $\kappa > 0$ and $\gamma \in [0, \pi]$ are physical constants. A profound explanation of the physical background together with a number of historical remarks can be found in Finn's monograph [1, Chapter 1]. The boundary value problem (1) and (2) goes back to Laplace, Young and Gauß and models the following physical situation, see also Fig. 1. A vertical capillary tube with horizontal cross section Ω is put into a huge container filled with liquid up to the zero-reference-level. Above the liquid, the space is filled with a specific gas. The solution $u : \overline{\Omega} \to \mathbb{R}$ describes in how far the capillary surface deviates inside the capillary tube from the zero reference

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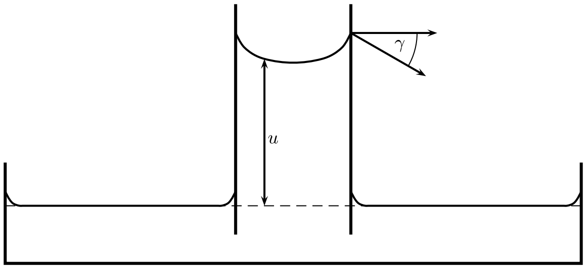

Fig. 1. Capillary surface.

level, which is also the equilibrium shape of the surface in case that the tube is not present. The capillarity constant $\kappa > 0$ depends on the interplay of surface tension (i.e. the interaction between liquid and gas) and gravity while the constant angle $\gamma \in [0, \pi]$ between the lower normal of the capillary surface and the exterior normal of the capillary tube depends on the interaction between the liquid, gas and the material of the tube. We observe that when u is a solution to (1) and (2), then $-u$ solves the same boundary value problem with γ replaced by $\pi - \gamma$. For this reason we may confine ourselves in what follows to $\gamma \in [0, \pi/2]$.

Since the differential equation (1) is not uniformly elliptic and the boundary condition (2) is (fully) nonlinear developing existence theories was somehow involved.

On the one hand one may first minimise the corresponding functional to obtain BV-solutions which under suitable conditions on Ω turn out to be smooth. In this context one has to mention authors like Emmer, Finn, Gerhardt, Giusti, Miranda (see e.g. [2–7]), for further references and a comprehensive exposition one may also see Finn's monograph [1, Chapter 7].

Here, however, we shall use the theory based on a priori estimates in classical Hölder spaces and fixed point arguments. According to works of Simon, Spruck, and Lieberman [8–10], in bounded C^4-domains Ω one has a unique solution $u \in C^{3,\alpha}(\overline{\Omega}) \cap C^\infty(\Omega)$ for some suitable $\alpha \in (0, 1)$ in the case $\gamma \in (0, \pi/2]$. If $\gamma = 0$ one has a solution $u \in C^0(\overline{\Omega}) \cap C^\infty(\Omega)$ in the sense that u is the uniform limit of smooth solutions u_γ for $\gamma \searrow 0$. The seemingly unnatural strong regularity requirements on the domain Ω are due to the fully nonlinear character of the boundary conditions. Uniqueness of these solutions is immediate thanks to a suitable comparison principle, see [1, Chapter 5]. One should observe that in the case $\gamma = \pi/2$ one has the trivial solution $u(x) \equiv 0$ so that this case need not be considered in what follows. Moreover, as discussed in [1, Chapter 5], those comparison principles allow for a number of interesting results on geometric, qualitative and quantitative properties of solutions.

In this respect, a particularly interesting and still widely open question concerns the possible convexity of solutions. Brulois [11] considered the case of Ω a disk and proved that for any $\gamma \in [0, \pi/2)$ these (due to uniqueness) radially symmetric solutions are uniformly convex, see Lemma 11. Korevaar [12] proved convexity of solutions for strictly convex C^1-smooth domains under the extreme boundary condition $\gamma = 0$, i.e. infinite slope at the boundary $\partial\Omega$. On the other hand, if $\gamma \in (0, \pi/2)$, there exist smooth convex domains Ω such that $\partial\Omega$ contains an edge with a sufficiently small angle, which in a suitable way is smoothly rounded over, where the corresponding solution is no longer convex. See [12, Theorem 2.4] and also [1], Theorem 5.5 and the remark at the end of Chapter 5.5. One would expect convexity of solutions if, depending on the boundary angle $\gamma \in (0, \pi/2)$, the boundary curvature of the uniformly convex domain does not deviate too much from a constant. Our main result is a first step in this direction. It shows that the convexity property of the radial solution in any disk persists under sufficiently small C^4-domain perturbations.

Theorem 1. *Suppose that $\kappa > 0$ and $\gamma \in (0, \pi/2)$. We consider a disk $\Omega := B_R(x_0)$ and for some small $\mu > 0$, a neighbourhood $\mathcal{U} := B_{R+\mu}(x_0)$. Then there exists an $\varepsilon = \varepsilon(R, \kappa, \gamma) > 0$ such that the following holds true:*

Consider any C^4-diffeomorphism $\Phi : \mathcal{U} \to \Phi(\mathcal{U})$ and $\Omega_\Phi := \Phi(B_R(x_0)) \subset \mathbb{R}^2$ with exterior unit normal $\nu_\Phi : \partial\Omega_\Phi \to \mathbb{S}^1$ such that $\|\Phi - \mathrm{Id}\|_{C^4(\overline{\Omega})} < \varepsilon$. Then the solution $u_\Phi \in C^3(\overline{\Omega}_\Phi)$ of the capillarity boundary value problem

$$\operatorname{div}\left(\frac{\nabla u_\Phi}{\sqrt{1+|\nabla u_\Phi|^2}}\right) = \kappa u_\Phi, \quad \text{in } \Omega_\Phi, \qquad \frac{\nabla u_\Phi \cdot \nu_\Phi}{\sqrt{1+|\nabla u_\Phi|^2}} = \cos(\gamma), \quad \text{on } \partial\Omega_\Phi,$$

is uniformly convex.

The first key observation in proving this result is that the classical $C^{1,\alpha}$-estimates (see Spruck's work [10] and references therein) are uniform for families of simply connected C^4-smooth domains as long as they obey the same quantitative bounds specified in Proposition 8. It is explained in Section 2 that all the underlying computations are in principle constructive and the estimation constants are in principle explicit.

Then, using an argument of Ladyzhenskaya and Ural'tseva (see [13, Chapter 10, pp. 463–465]) we show by means of a proof by contradiction a $C^{2,\beta}$-stability result for the solutions with respect to small C^4-domain perturbations. See Theorem 9. In view of the uniform convexity of radial solutions in balls due to Brulois (see [11] and also Lemma 11) this yields the proof of Theorem 1 which is given in Section 4.

Most of our notation is quite standard and mostly adopted from [14].