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In any dimension a "clamped plate" with a uniform weight may change sign☆

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ABSTRACT

Positivity preserving properties have been conjectured for the bilaplace Dirichlet problem in many versions. In this note we show that in any dimension there exist bounded smooth domains Ω such that even the solution of $\Delta^2 u = 1$ in Ω with the homogeneous Dirichlet boundary conditions $u = u_\nu = 0$ on $\partial\Omega$ is sign-changing. In two dimensions this corresponds to the Kirchhoff–Love model of a clamped plate with a uniform weight.

1. Introduction

It is well known that for bounded smooth domains $\Omega \subset \mathbb{R}^n$ with outside unit normal ν, the biharmonic boundary value problem

$$\begin{cases} \Delta^2 u = f & \text{in } \Omega, \\ u = \dfrac{\partial}{\partial \nu} u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

is in general not sign preserving unless the domain is a ball or close to a ball, see [1–3]. In these papers it was shown that the corresponding Green function is positive, which is equivalent with (1) being sign preserving. A first counterexample, which shows that (1) is not positivity preserving on arbitrary domains, is due to Duffin in [4], cf. also [5]. The most striking one, showing sign change of u with a suitable $f \geq 0$ with $\Omega \subset \mathbb{R}^2$ being a mildly eccentric ellipse, was found by Garabedian, see [6]. For a short history of this problem we refer to [7]. The weaker question, whether or not the first eigenfunction is of one sign, has been studied e.g. in [8–10]. For an overview see also [11]. Although a wider class of domains are allowed for this eigenfunction to be of one sign, on general domains the fixed sign cannot be expected. Some questions on how the sign change of both problems are related are found in [12].

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In the present note we consider the apparently still weaker question, whether or not the solution of

$$\begin{cases} \Delta^2 u = 1 & \text{in } \Omega, \\ u = \dfrac{\partial}{\partial \nu} u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2}$$

which is (1) with $f = 1$, is positive. This question was raised by Svitlana Mayboroda and for a motivation from an applied point of view see [13]. In the previous note [14] we constructed a counterexample in $\mathbb{R}^2$, which is based on Garabedian's celebrated example [6]. In that note we use the inversion as a particular Möbius transformation and corresponding covariance properties of the biharmonic operator. This note will show by means of an inductive procedure that sign change may occur in any dimension. This generalises and simplifies an approach by Nakai and Sario in [15].

The precise statement of our main result is as follows.

Theorem 1. *For any integer $n \geq 2$, there are bounded smooth domains $\Omega \subset \mathbb{R}^n$ such that the solution u of (2) changes sign.*

2. An inductive procedure

In [14, Theorem 2.4] one finds the following result:

- *There are bounded C^∞-smooth domains $\Omega \subset \mathbb{R}^2$ such that the solution of (2) changes sign.*

The proof is based on the fact that a solution u of (2) composed with an inversion $h(x) = |x|^{-2} x$ with $0 \notin \Omega$ satisfies

$$\Delta^2 \left(u \circ h(x)\right) = |x|^{-6}. \tag{3}$$

One takes a domain Ω for which the Green function changes sign near opposite boundary points and moves Ω such that the centre of inversion 0 is located outside of Ω but near one such a boundary point. The final step consists of showing that the singularity in (3) is sufficiently close to a δ-distribution near the first boundary point in order to keep the negative sign near the opposite boundary point. See [14] for details.

With [14, Theorem 2.4] it suffices to show the following.

Theorem 2. *Let $n \geq 2$. Assume that there is a bounded smooth domain $A \subset \mathbb{R}^n$ for which the solution of (2) with $\Omega = A$ is sign-changing. Then there exists a bounded smooth domain $A^* \subset \mathbb{R}^{n+1}$ for which the solution of (2) with $\Omega = A^*$ is sign-changing.*

In order to prove this result we pick a dimension $n \geq 2$ and a bounded smooth domain $A \subset \mathbb{R}^n$ and assume that the corresponding smooth solution $u : \bar{A} \to \mathbb{R}$ of (2) is sign changing. Writing $x = (x', x_{n+1}) \in \mathbb{R}^{n+1}$ and putting

$$A^*_\infty := A \times \mathbb{R}, \qquad u_\infty(x', x_{n+1}) := u(x'), \tag{4}$$

we immediately get a sign changing solution of (2) in the unbounded cylindrical domain $A^*_\infty \subset \mathbb{R}^{n+1}$. The idea is to suitably cap off A^*_∞ to a bounded smooth domain A^*_h. See Fig. 1. We solve (2) for these bounded domains and will show that the corresponding solution is still sign changing when h is large enough.

We start with a technical result.

Lemma 3. *Let $A \subset \mathbb{R}^n$ be a smooth and bounded domain. Then there exists a function $g_A \in C^0(\bar{A}, [0, 1]) \cap C^\infty(A, [0, 1])$ such that for any $h > 0$ the domains $A^*_h \subset \mathbb{R}^{n+1}$, defined by*

$$A^*_h := \left\{ \left(x', x_{n+1}\right) : x \in A, \ \ -h - g_A(x') < x_{n+1} < h + g_A(x') \right\} \tag{5}$$

are smooth.

Proof. The signed distance $d\,(\partial A, \cdot) : \mathbb{R}^n \to \mathbb{R}$ to the boundary of A is defined by

$$d\left(\partial A, x'\right) = \begin{cases} \inf\left\{\left|x' - \tilde{x}\right| ; \tilde{x} \in \partial A\right\} & \text{for } x' \in \bar{A}, \\ -\inf\left\{\left|x' - \tilde{x}\right| ; \tilde{x} \in \partial A\right\} & \text{for } x' \notin \bar{A}. \end{cases}$$

Since ∂A is smooth and bounded, there exists $r_A \in (0, 1)$, such that A satisfies a uniform interior sphere condition as well as a uniform exterior sphere condition both with spheres of radius r_A. Moreover, the function $d(\partial A, \cdot)$ is smooth on $\partial A + B_{r_A}(0)$. See [16]. Let $f \in C^\infty(\mathbb{R})$ be nondecreasing such that

$$f(s) = \begin{cases} -\dfrac{1}{2} & \text{for } s < -\dfrac{2}{3}, \\ s & \text{for } s \in \left[-\dfrac{1}{3}, \dfrac{1}{3}\right], \\ \dfrac{1}{2} & \text{for } s > \dfrac{2}{3}, \end{cases}$$