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Factorization theorems of Arendt type for additive monotone mappings

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ABSTRACT

We deal with additive monotone mappings defined on a lattice-ordered Abelian group and having values in a Dedekind complete Riesz space and which are invariant with respect to some representation of an amenable semigroup. Using a Hahn–Banach-type theorem of Zbigniew Gajda, we obtain generalizations of factorization theorems obtained in 1984 by Wolfgang Arendt for positive linear operators. The theorems of Arendt are generalized in two directions. First, we extend these results from the case of linear operators acting between Riesz spaces to the case of additive mappings between lattice-ordered Abelian groups. Second, we study mappings which are invariant with respect to a semigroup representation.

As an application of the results obtained, we show some property of composition operators between spaces of additive functions acting between lattice-ordered groups.

1. Introduction

The motivation for the present studies comes from abstract versions of the Radon–Nikodym theorem, which were obtained first by Maharam [1,2]. She dealt with function-valued measures and abstract integrals. Later, her results were generalized in the framework of Riesz spaces by Luxemburg and Schep [3, Theorems 3.1, 3.4 and 4.2] and by Huijsmans and Luxemburg [4, Theorem 0.7], and others.

Let F and G be two Riesz spaces, and let $U: G \to F$ be a positive linear operator. Then U is said to have the *Maharam property* if for all $g \in G$ and for all $f \in F$ such that $g \geq 0$ and $0 \leq f \leq Ug$ there exists some $g_1 \in G$ such that $0 \leq g_1 \leq g$ and $Ug_1 = f$ (see [3, Section 2]).

The Luxemburg–Schep theorem [3, Theorem 3.1] says that, if the Riesz spaces F and G are Dedekind complete and the operator $U: G \to F$ is order continuous, then the Maharam property of U is equivalent to the following fact, which is an operator version of the assertion of the Radon–Nikodym theorem.

For every operator $T: G \to F$ such that $0 \leq T \leq U$ there exists an orthomorphism π of G such that $0 \leq \pi \leq I$ and $T = U \circ \pi$.

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Recall that an orthomorphism on a Riesz space is an order-bounded linear mapping such that $f \perp g$ implies that $\pi f \perp g$ (the symbol $\perp$ denotes the disjointness of elements; i.e., $f \perp g$ if $|f| \wedge |g| = 0$). A typical example of an orthomorphism is the operator of multiplication:

$$(\pi f)(x) \mapsto g(x)f(x),$$

defined on some space, for example on $L^p(X, \mu)$, with g being μ-measurable and μ-almost everywhere finite. Therefore, the orthomorphism π in the Luxemburg–Schep theorem plays the role of the Radon–Nikodym derivative in the classical Radon–Nikodym theorem.

A dual version of the operator Radon–Nikodym theorem is also proved in [3]. For given positive linear mappings T and V defined on an Archimedean Riesz space F and having values in a Dedekind complete Riesz space G, the existence of an orthomorphism π of G such that $T = \pi \circ V$ is characterized in [3, Theorem 4.2].

In 1984, Arendt [5] proved the following two theorems.

Theorem 1 (*Arendt [5, Theorem 1.1]*)**.** *Let E be a Dedekind complete Riesz space, let F and G be Riesz spaces, and let $V : F \to G$ be a Riesz homomorphism. Then, given a positive linear mapping $S : G \to E$, every positive linear mapping $T : F \to E$ which satisfies $T \leq S \circ V$ admits a factorization*

$$T = S_1 \circ V,$$

where $S_1 : G \to E$ is a linear mapping such that $0 \leq S_1 \leq S$.

Theorem 2 (*Arendt [5, Theorem 1.4]*)**.** *Let E, F, and G be Banach lattices with G having order-continuous norm, and let $U : G \to F$ be an interval-preserving positive linear mapping. Then, given a positive linear mapping $S : E \to G$, every positive linear mapping $T : E \to F$ which satisfies $T \leq U \circ S$ admits a factorization*

$$T = U \circ S_1,$$

where $S_1 : E \to G$ is a linear mapping such that $0 \leq S_1 \leq S$.

A particular case of the second Arendt theorem, for $G = E$ and with S being equal to the identity mapping, coincides with the Luxemburg–Schep theorem. Notice that the Maharam property is replaced by the interval-preserving property. The first Arendt theorem generalizes the dual Luxemburg–Schep theorem in a similar way. It is also worth mentioning that the proofs of the Arendt theorems are relatively short in comparison with the original proofs of Maharan's results and the Luxemburg–Schep theorem.

In the paper we will join the original approach of Arendt from his factorization theorems (Theorems 1 and 2) with the concept of lattice-ordered groups introduced by Birkhoff [6] and with the idea of Gajda of using amenable semigroup techniques to study functional equations with solutions which are invariant with respect to a semigroup representation (see [7,8]).

We obtain extensions of the Arendt theorems in a more general framework, and also we investigate operators which are invariant under an amenable semigroup representation. We will deal with mappings defined on lattice-ordered groups (ℓ-groups for short) which are invariant with respect to some representation of an amenable semigroup. Therefore, we will consider monotone (order-preserving) additive mappings instead of linear operators, and homomorphisms of ℓ-groups instead of Riesz homomorphisms.

Let $(X, \cdot)$ denote a right-amenable semigroup, and let $G = (G, +, \leq)$ be a partially ordered Abelian group. Further, let $\Phi : X \to \mathrm{End}(G)$ be a representation of X in the semigroup $\mathrm{End}(G)$ of endomorphisms of G. We will preserve the usual convention and we will write Φ_s instead of $\Phi(s)$ for $s \in X$. Therefore, the following equality holds true:

$$\Phi_{st} = \Phi_s \circ \Phi_t, \quad s, t \in X.$$

If X is a group, then also

$$\Phi_{s^{-1}} = (\Phi_s)^{-1}, \quad s \in X;$$

in particular, every Φ_s is an invertible map.

Let F be another partially ordered group. A map $f : G \to F$ is called *monotone* if

$$x \leq y \Longrightarrow f(x) \leq f(y)$$

for all $x, y \in G$, and f is called *subadditive* if

$$f(x+y) \leq f(x) + f(y)$$

for all $x, y \in G$. Further, f is called *Φ-subinvariant* if $f \circ \Phi_s \leq f$ for all $s \in X$, and f is *Φ-invariant* if $f \circ \Phi_s = f$ for all $s \in X$.

If G and F are ℓ-groups, then by homomorphism of ℓ-groups we mean every additive homomorphism of lattices $f : G \to F$. It is straightforward to see that every such mapping is in particular monotone. To avoid confusion we will be using the term additive mapping instead of homomorphism whenever we mean a group homomorphism between ℓ-groups which is not an ℓ-group homomorphism.

Let G and F be Abelian ℓ-groups. The following easy property of odd monotone mappings will be used later.

Proposition 1. *Assume that $f : G \to F$ is an odd monotone mapping. Then $|f(x)| \leq f(|x|)$ for all $x \in G$.*