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Phragmén–Lindelöf theorems for equations with nonstandard growth

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ABSTRACT

The Phragmén–Lindelöf theorem on unbounded domains is studied for subsolutions of variable exponent $p(\cdot)$-Laplace equations of homogeneous and nonhomogeneous types. The discussion is illustrated by a number of examples of unbounded domains such as half space, angular domains and domains narrowing at infinity. Our approach gives some new results also in the setting of the p-Laplacian and the harmonic operator.

1. Introduction

In this paper we study the growth of $p(\cdot)$-harmonic subsolutions on unbounded domains in $\mathbb{R}^n$. Let u be a local weak subsolution in an unbounded domain Ω of either

$$\operatorname{div}(|\nabla u|^{p(\cdot)-2}\nabla u) = 0 \tag{1.1}$$

or

$$\operatorname{div}(|\nabla u|^{p(\cdot)-2}\nabla u) = f(x, u, \nabla u),$$

under suitable assumptions on function f. For solutions of such equations we investigate the asymptotic behavior of u in $\Omega \cap B_R$ for large radii R, where B_R denotes the ball of radius R centered at the origin. The prototype for our studies is the following classical Phragmén–Lindelöf theorem in the plane [1].

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Let u be subharmonic in the upper half plane and let $\lim_{z\to\mathbb{R}^+} u(z) \le 0$. *Then either* $u \le 0$ *in the whole upper plane or it holds that*

$$\liminf_{R\to\infty} \frac{\sup\{u(z) : |z| = R\}}{R} > 0.$$

This result was extended to the setting of elliptic equations of second order in [2,3], has been studied for elliptic equations in general domains [4], fully nonlinear equations [5,6], as well as in the context of Riemannian manifolds [7], see also [8,9] for some further generalizations of the Phragmén–Lindelöf alternative. As for relation to applied sciences let us mention that the Phragmén–Lindelöf principle is connected to the so-called Saint-Venant's Principle in elasticity theory (for more details see e.g. [10]).

One of the most fundamental equations of nonlinear analysis is the p-harmonic equation:

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0 \quad 1 \le p \le \infty.$$

The importance of this equation comes among others from the fact that it is a natural nonlinear generalization of harmonic functions ($p = 2$), has variational characterization in terms of p-Dirichlet energy; also appears in numerous areas of pure and applied mathematics to mention for example differential geometry, viscosity solutions (especially the case $p = \infty$), relation to quasiregular mappings, nonlinear eigenvalue problems. One also studies generalizations of p-harmonic functions on metric spaces. As for applied sciences the p-Laplace equation is used as a model equation in nonlinear elasticity theory, glaciology, stellar dynamics, and description of flows through porous media.

Another recently blooming area in nonlinear analysis is the theory of PDEs with nonstandard growth (variable exponent analysis) and related energy functionals. Eq. (1.1) serves as the model example. Here p is a measurable function $p : \Omega \to [1, \infty]$ called the variable exponent while solutions naturally belong to the appropriate Musielak–Orlicz space (see Preliminaries). Apart from interesting theoretical considerations such equations naturally arise, for instance, as a model for thermistor [11], in fluid dynamics [12], in the study of image processing [13] and electro-rheological fluids [14]; see [15] for a recent survey and further references, see also the monograph [16], where the role of (non)homogeneous $p(\cdot)$-Laplace equations in applications is discussed in more detail. Despite the symbolic similarity to the constant exponent equations, various unexpected phenomena may occur when the exponent is a function, for instance the minimum of the $p(\cdot)$-Dirichlet energy may not exist even in the one-dimensional case for smooth functions p; also smooth functions need not be dense in the corresponding variable exponent Sobolev spaces.

Several features of Eq. (1.1) have been studied, for example the regularity theory, potential theory, Harnack type estimates and boundary regularity to mention just a few (see [15] and references therein). Such an equation has, however, many disadvantages comparing to the $p =$ const case, for instance: lack of scalability of solutions, nonhomogeneous Harnack inequalities with constant depending on the solution. These often make the analysis of the nonstandard growth equation difficult and lead to technical and nontrivial estimates (nevertheless, see [17,18] and Remark 3.4 below for a variant of Eq. (1.1) that overcomes some of the described difficulties, the so-called strong $p(\cdot)$-harmonic equation).

We would like now to discuss the state of art for the problem in the case of the Phragmén–Lindelöf principle for p-Laplacian and explain some difficulties arising when extending known approaches to the variable exponent setting. Lindqvist in [19] proved the principle for special domains of type $\mathbb{R}^n \setminus H^q$, where H^q is a q-dimensional hyperplane. This approach relies on n-harmonic measures and the comparison principle. Unfortunately, the same technique cannot be applied in our setting due to the lack of scalability for the $p(\cdot)$-harmonic equation and lack of similar relations between n-harmonic measures and $p(\cdot)$-harmonic operators. Nevertheless, by using our approach, in Corollary 3.5 we retrieve part of Theorem 4.6 in [19] as a special case of one of our main results, Theorem 3.3. Another interesting approach toward the Phragmén–Lindelöf principle was taken by Granlund [20] and is based on de Giorgi type estimates and their iterations. The corresponding estimates for the $p(\cdot)$-harmonic operator are non-homogeneous and their iterations do not lead to the desired result as in [20]. Results by Jin and Lancaster discussed in [8], although applicable to wide class of quasilinear elliptic equations with C^2 solutions, cannot be directly used in our setting as the $p(\cdot)$-harmonic functions are, in general, $C^{1,\alpha}$ regular (cf. [21]). As for p-harmonic equations with nontrivial right-hand side we mention work of Kurta [22], where the Phragmén–Lindelöf theorem is proven for $|\nabla u|$ together with existence results for nontrivial solutions (see also [23]).

Organization of the paper

In Section 2 we recall basic facts and properties of variable exponent spaces, variational capacities and $p(\cdot)$-harmonic functions.

Section 3 is devoted to studying the main result of the paper, namely the Phragmén–Lindelöf theorem for subsolutions of the homogeneous $p(\cdot)$-harmonic equation. Our approach is based on developing an energy estimate for the norm of the gradient of a $p(\cdot)$-harmonic subsolution. Such estimate carries information about: (a) the impact of the rate of growth of variable exponent $p(\cdot)$; (b) the size of the underlying domain expressed in terms of capacity; (c) the porosity of the domain. Under growth assumptions on the exponent we provide a general condition implying the assertion of the theorem and illustrate discussion by a number of corollaries for domains typically appearing in the context of the Phragmén–Lindelöf alternative: a half space, an angular sector, a domain narrowing at infinity.

In Section 4 we present the corresponding results for a nonhomogeneous $p(\cdot)$-harmonic equation. Our approach gives some new results also in the setting of p-Laplacian and harmonic functions, see Corollaries 4.4 and 4.5.