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Existence and decay of solutions for a viscoelastic wave equation with acoustic boundary conditions

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ABSTRACT

In this paper we consider a multi-dimensional damped semilinear viscoelastic wave equation with variable coefficient and acoustic boundary conditions. First, we prove a local existence theorem by using the Faedo–Galerkin approximations combined with a contraction mapping theorem. Second, we show that, under suitable conditions on the initial data and the relaxation function, the solution exists globally in time and we prove the uniform decay rate of the energy.

1. Introduction

Let Ω be a bounded domain of $\mathbb{R}^N$, $N \geq 1$, with a smooth boundary $\Gamma = \Gamma_0 \cup \Gamma_1$. Here Γ_0 and Γ_1 are closed and disjoint and $\nu = (\nu_1, \ldots, \nu_n)$ represents the unit outward normal to Γ. In this work, we are concerned with the local, uniqueness, global solution and the decay of energy solution of the following model

$$\begin{cases} u_{tt} + Lu - \displaystyle\int_0^t g(t-s)\, Lu(s)\, ds = |u|^{p-2} u, & \text{in } \Omega \times (0,\infty), \\ u = 0, & \text{on } \Gamma_0 \times (0,\infty), \\ \dfrac{\partial u}{\partial \nu_L} - \displaystyle\int_0^t g(t-s) \dfrac{\partial u}{\partial \nu_L}(s)\, ds = h(x) z_t, & \text{on } \Gamma_1 \times (0,\infty), \\ u_t + f(x) z_t + m(x) z = 0 & \text{on } \Gamma_1 \times (0,\infty), \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), & \text{in } \Omega, \\ z(x,0) = z_0(x), & \text{on } \Gamma_1, \end{cases} \tag{1.1}$$

where $Lu = -div\,(A\nabla u) = -\sum_{i,j=1}^N \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right)$ and $\frac{\partial u}{\partial \nu_L} = \sum_{i,j=1}^N a_{ij}(x)\frac{\partial u}{\partial x_j}\nu_i$. The functions $f,\ m,\ h$ are essentially bounded and $p > 2$. $u_0, u_1 : \Omega \to \mathbb{R}$ and $z_0 : \Gamma_1 \to \mathbb{R}$ are given functions. System (1.1) represents a viscoelastic wave

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equation with variable coefficients and acoustic boundary conditions which is a coupling of hyperbolic/parabolic equations, where the coupling is given on the portion Γ_1 of the boundary which is usually called the interface, while the complement Γ_0 of Γ_1 in Γ is called the hard wall. The part Γ_1 of the boundary exhibits some porosity, which is, see [1], given in the third and the fourth equations in (1.1).

The physical applications of the above system is related to the problem of noise control and suppression in practical applications. The noise sound propagates through some acoustic medium, for example, through air, in a room which is characterized by a bounded domain Ω and whose walls, ceiling and floor are described by the boundary conditions. This is the description of Jieqiong Wu in [2]. For a more physical explanation of wave equations with acoustic boundary conditions when $g = 0$, we refer the reader to [3–9].

The acoustic boundary conditions were introduced by Beale and Rosencrans in [4,10], where the authors proved the global existence and regularity of solutions in a Hilbert space of the linear problem:

$$\begin{cases} u_{tt} = \Delta u, & \text{in } \Omega \times (0, \infty), \\ \dfrac{\partial u}{\partial \nu} = z_t, & \text{on } \partial\Omega \times (0, \infty), \\ u_t + m(x) z_{tt} + p(x) z_t + q(x) z = 0, & \text{on } \partial\Omega \times (0, \infty), \end{cases} \tag{1.2}$$

where $m,\ p$ and q are nonnegative functions on the boundary with m and q being strictly positive. The solution $u(x, t)$ of the wave equation (first equation) of system (1.2) is the velocity potential of a fluid undergoing acoustic wave motion and $z(x, t)$ is the normal displacement to the boundary at time t with the boundary point x. In this situation the boundary is called a locally reacting boundary. See also the paper [11] for a related model. In addition, the author in [10] showed that when the second derivative z_{tt} was included in the third equation of (1.2) there was no uniform rate of decay of the energy associated.

Frota and Larkin [1] eliminated the term z_{tt} and established global solvability and decay estimates for a linear wave equation with boundary conditions

$$\begin{cases} u_{tt} - \Delta u + \alpha(x) u = 0, & \text{in } \Omega \times (0, \infty), \\ u = 0, & \text{on } \Gamma_0 \times (0, \infty) \end{cases} \tag{1.3}$$

with the same acoustic boundary conditions in system (1.1) (with $g \cong 0$). The physical significance of this problem is that the surface material is much lighter than a liquid flowing along it. It was observed that porous walls of channels or airfoils stabilize hydrodynamic flows, [1]. However, they said that is not simple to apply the Faedo–Galerkin method because the system of corresponding ordinary equations is not normal and cannot apply directly the Carathéodory theorem. To overcome this difficulty, they considered the third equation as a degenerated second order equation

$$u_t + \epsilon z_{tt} + f(x) z_t + m(x) z = 0, \quad \text{on } \Gamma_1 \times (0, \infty),$$

where $\epsilon \to 0$. This is the case of our problem and the problem of Park and Ha, in [12] they considered the wave equation with acoustic boundary conditions. That is system (1.1) with the first equation is replaced by

$$u_{tt} - \Delta u + |u_t|^p u_t + |u|^q u = 0, \quad \text{in } \Omega \times (0, \infty),$$

and proved the existence of global solutions and uniform decay rates of energy.

Using the nonlinear semigroup theory, Graber in [13] generalized and proved the existence and uniqueness of local solution of the problem of Frota and Larkin, where the first acoustic conditions is replaced by

$$\frac{\partial u}{\partial \nu} = h(x)\, \eta(z_t), \quad \text{on } \Gamma_1 \times (0, \infty),$$

where η is a nonlinear function. Recently, the same author and Said-Houari considered in [14] a semilinear wave equation with semilinear porous acoustic boundary conditions with nonlinear boundary/interior sources and a nonlinear boundary/interior damping. They showed the existence and uniqueness of local in time solution, by using the nonlinear semigroup theory. Then, in different cases, they proved the decay rates of the solution, the blow up in finite time and they proved that the solution is unbounded and grows as an exponential function in finite time.

In the case: $L = -\Delta$ and absence of the source term $|u|^{p-2} u$, problem (1.1) has been investigated in [8], where the authors proved some decay rates under the assumption that the kernel g is assumed to decay like $g'(t) \le -\xi(t) g(t)$ where ξ is a nonincreasing and positive function and satisfies some technical conditions and

$$l = 1 - \int_0^\infty g(s)\, ds < 1/2. \tag{1.4}$$

The main idea in the proof of [8] is essentially based on the work in [15] where the author proved the same result for the viscoelastic wave equations with Dirichlet boundary conditions on the whole boundary $\partial\Omega$. It is by now well known that the decay of the solutions of (1.1) strongly depends on the decay of the kernel g. Viscoelastic wave equations have been studied by many authors. For example, Berrimi and Messaoudi in [16] considered the following problem with Dirichlet boundary conditions,

$$u_{tt} - \Delta u + \int_0^t g(t-s)\, \Delta u(s)\, ds = |u|^{p-2} u, \quad \text{in } \Omega.$$