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A note on lower bounds of decay rates for solutions to the Navier–Stokes equations in the norms of Besov spaces

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ABSTRACT

We study the lower bounds of decay rates for turbulent solutions to the Navier–Stokes equations in the norms of Besov spaces. We focus on solutions satisfying $0 < \liminf_{t\to\infty} t^\gamma\|u(t)\| \leq \limsup_{t\to\infty} t^\gamma\|u(t)\| < \infty$ for some $\gamma \in (0, 5/4]$ and prove among others that such solutions, measured in the norm of the Besov space $\dot{B}^{-2\gamma}_{2,\infty}$, are estimated for large times from below by some positive constant. These estimates stem from sufficiently fast rate of large time energy concentration in low frequencies occurring in the studied solutions. Our results have simple proofs and improve and extend substantially the results published by Miyakawa (2002) and by Schonbek and Wiegner (1996).

1. Introduction

Consider the Navier–Stokes equations in $\mathbf{R}^3$:

$$\frac{\partial u}{\partial t} - \Delta u + u\cdot\nabla u + \nabla p = 0 \quad \text{in } \mathbf{R}^3 \times (0,\infty), \tag{1}$$

$$\nabla\cdot u = 0 \quad \text{in } \mathbf{R}^3 \times (0,\infty), \tag{2}$$

$$u|_{t=0} = u_0, \tag{3}$$

where $u = u(x,t) = (u_1(x,t), u_2(x,t), u_3(x,t))$ and $p = p(x,t)$ denote the unknown velocity and pressure and $u_0 = u_0(x) = (u_{01}(x), u_{02}(x), u_{03}(x))$ is a given initial velocity.

In the paper we will study the lower bounds of decay rates for turbulent solutions to (1)–(3) in the norms of the Besov spaces with negative derivative indices. If $u_0 \in L^2_\sigma$ (for the notation see the next section) then a measurable function u defined on $\mathbf{R}^3 \times (0,\infty)$ is called a turbulent solution to (1)–(3) (see [1]) if $u \in L^\infty((0,\infty); L^2_\sigma) \cap L^2_{loc}([0,\infty); \dot{H}^1)$, it satisfies (1) in the sense of distributions and it also satisfies the strong energy inequality

$$\|u(t)\|^2 + 2\int_s^t \|\nabla u(\sigma)\|^2 d\sigma \leq \|u(s)\|^2$$

for $s = 0$ and almost all $s > 0$, and all $t \geq s$. If $u_0 \in L^2_\sigma$ then there exists at least one turbulent solution to (1)–(3) (see [2]). We will study the turbulent solutions satisfying

$$0 < \liminf_{t\to\infty} t^\gamma\|u(t)\| \leq \limsup_{t\to\infty} t^\gamma\|u(t)\| < \infty \tag{4}$$

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for some $\gamma \in (0, 5/4]$ (their existence will be discussed further in this section). Note here that generic turbulent solutions exhibit the large time energy concentration in frequencies from contracting balls with the center at (0, 0, 0) in the frequency space, precisely (see [3])

$$\lim_{t\to\infty} \|E_\varepsilon u(t)\|/\|u(t)\| = 1$$

for every $\varepsilon > 0$. We will show that the solutions satisfying (4) exhibit the large time energy concentration in frequencies from contracting annuli with the center at (0, 0, 0) in the frequency space. As we will see in Section 3, this phenomenon is connected with the existence of the lower bounds of decay rates for these solutions in the norms of the Besov spaces with negative derivative indices. We prove as our main result the following theorem:

Theorem 1. *Let u be a turbulent solution to (1)–(3) satisfying the inequalities (4). Let $\kappa \in (0, 1)$. Then there exist $\beta, \tilde\beta > 0$ such that $\liminf_{t\to\infty} \|(E_{\mu(t)} - E_{\tilde\mu(t)})u(t)\|/\|u(t)\| \geq \kappa$, where $\mu(t) = \beta t^{-1}$ and $\tilde\mu(t) = \tilde\beta t^{-1}$. Further, if $\alpha \geq 0$, then*

$$\liminf_{t\to\infty} t^{\alpha/2}\|u(t)\|_{\dot B^{-2\gamma+\alpha}_{2,\infty}} > 0. \tag{5}$$

Remark 1. The inequality from Theorem 1, $\liminf_{t\to\infty} \|(E_{\mu(t)}-E_{\tilde\mu(t)})u(t)\|/\|u(t)\| \geq \kappa$, can be replaced due to Proposition 1 (see below) by the inequality

$$\liminf_{t\to\infty}\left(\int_{B_{\sqrt{\mu(t)}}(0)\setminus B_{\sqrt{\tilde\mu(t)}}(0)} |\mathcal{F}(u(t))(\xi)|^2\,d\xi \Big/ \int |\mathcal{F}(u(t))(\xi)|^2\,d\xi\right) \geq \kappa^2.$$

Remark 2. It follows from Theorem 2 below that $u(t) \in \dot B^{-2\gamma}_{2,\infty}$ for every $t \geq 0$. Further, $u(t) \in \dot H^s = \dot B^s_{2,2}$ for every $s \geq 0$ and all large t (see [4]). It follows from the definition of the Besov spaces (see the next section) that $u(t) \in \dot B^{-2\gamma+\alpha}_{2,\infty}$ for all sufficiently large t and the left hand side of (5) is well defined.

The basic idea of the proof of Theorem 1 is the following one: in solutions with energy concentrating for large times sufficiently quickly in low frequencies one can expect the existence of lower bounds of decay rates in the norms of the Besov spaces with negative derivative indices—see the definition of these spaces in the next section.

We discuss now the existence of turbulent solutions satisfying (4). The turbulent solutions satisfying $\limsup_{t\to\infty} t^\gamma\|u(t)\| < \infty$ are characterized by their initial conditions belonging to $\dot B^{-2\gamma}_{2,\infty}$, as follows from the following theorem (for its proof see [5]):

Theorem 2. *Suppose that $u_0 \in L^2_\sigma$, u is a turbulent solution of (1)–(3) and $\gamma \in [0, 5/4]$. Then the following conditions are equivalent:*

(i) $\limsup_{t\to\infty} t^\gamma\|u(t)\| < \infty$,
(ii) $u(t_0) \in \dot B^{-2\gamma}_{2,\infty}$ *for every* $t_0 \geq 0$,
(iii) $u(t_0) \in \dot B^{-2\gamma}_{2,\infty}$ *for some* $t_0 \geq 0$,
(iv) $u_0 \in \dot B^{-2\gamma}_{2,\infty}$.

If these conditions hold then there exists $c > 0$ such that $\|u(t_0)\|_{\dot B^{-2\gamma}_{2,\infty}} \leq c$ for every $t_0 \geq 0$ and $\|u(t)\| \leq \tilde c(1+t)^{-\gamma}$ for every $t \geq 0$, $\tilde c$ depending only on $\|u_0\|_{\dot B^{-2\gamma}_{2,\infty}}$ and $\|u_0\|$.

Further, Wiegner proved in [6] that if for $u_0 \in L^2_\sigma$ $\limsup_{t\to\infty} t^\gamma\|e^{\Delta t}u_0\| < \infty$ for some $\gamma \in (0, 5/4]$, then $\limsup_{t\to\infty} t^\gamma\|u(t)\| < \infty$ and moreover, if $\gamma \in (0, 5/4)$, then $\lim_{t\to\infty} t^\gamma\|u(t) - e^{\Delta t}u_0\| = 0$, where u is a turbulent solution with the initial condition u_0. Consequently, a turbulent solution u satisfies (4) for some $\gamma \in (0, 5/4)$ if and only if for its initial condition $u(0) = u_0$

$$0 < \liminf_{t\to\infty} t^\gamma\|e^{\Delta t}u_0\| \leq \limsup_{t\to\infty} t^\gamma\|e^{\Delta t}u_0\| < \infty. \tag{6}$$

There is no problem with the inequality $\limsup_{t\to\infty} t^\gamma\|e^{\Delta t}u_0\| < \infty$. It is well known (see [7] or [8]) that this inequality is satisfied if and only if $u_0 \in \dot B^{-2\gamma}_{2,\infty}$. In the following lemma we present a simple example of initial conditions satisfying also the inequality $0 < \liminf_{t\to\infty} t^\gamma\|e^{\Delta t}u_0\|$.

Lemma 1. *Let $0 < \alpha_1 \leq \alpha_2$, $\delta > 0$, $m > -3/2$ and*

$$u_0 \in K^\delta_{m,\alpha_1,\alpha_2} = \{v \in L^2_\sigma;\ \alpha_1|\xi|^m \leq |\mathcal{F}(v)(\xi)| \leq \alpha_2|\xi|^m,\ \forall|\xi| \leq \delta\}.$$

Then

$$0 < \liminf_{t\to\infty} t^{(m+3/2)/2}\|e^{\Delta t}u_0\| \leq \limsup_{t\to\infty} t^{(m+3/2)/2}\|e^{\Delta t}u_0\| < \infty.$$