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On the Cauchy problem for a general fractional porous medium equation with variable density

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ABSTRACT

We study the well-posedness of the Cauchy problem for a fractional porous medium equation with a varying density $\rho > 0$. We establish existence of weak energy solutions; uniqueness and nonuniqueness is studied as well, according to the behavior of ρ at infinity.

1. Introduction

In this paper we are concerned with nonnegative solutions to the following nonlinear nonlocal Cauchy problem:

$$\begin{cases} \rho\, \partial_t u + (-\Delta)^{\frac{\sigma}{2}}\left[u^m\right] = 0 & x \in \mathbb{R}^N,\ t > 0 \\ u = u_0 & x \in \mathbb{R}^N,\ t = 0. \end{cases} \tag{1.1}$$

The nonlocal operator $(-\Delta)^{\frac{\sigma}{2}}$ is the fractional Laplacian of order $\sigma/2$; see for instance [1,2] for a comprehensive account on the subject. The parameter σ is supposed to vary in the open interval $(0, 2)$, thus a representation for such an operator in terms of a singular integral holds. The function $\rho(x)$ is a density; it is assumed to be positive and to depend continuously on the spatial variable x. The initial value u_0 is a bounded function belonging to the weighted space $L^1_\rho(\mathbb{R}^N)$ of measurable functions f satisfying $\int_{\mathbb{R}^N} f\, \rho\, \mathrm{d}x < \infty$. Finally, $N \geq 1$ and m is a real parameter greater or equal to 1. The aim of this paper is to investigate existence and uniqueness of solutions to problem (1.1).

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By replacing the nonlocal operator in (1.1) with the classical Laplace operator Δ we obtain the initial value problem for the porous medium equation with variable density:

$$\begin{cases}\rho\,\partial_t u - \Delta u^m = 0 & x \in \mathbb{R}^N,\ t > 0\\ u = u_0 & x \in \mathbb{R}^N,\ t = 0.\end{cases} \tag{1.2}$$

Problem (1.2) have been extensively studied in the literature; see [3–11], and also [12] where similar problems on Riemannian manifolds have been taken into account. The picture for problem (1.2) has been completed in [10] where existence and uniqueness of solutions to this problem have been established in the class of finite energy solutions assuming the initial data u_0 is nonnegative and in L^1_ρ. Uniqueness of solutions to (1.2) is a delicate issue and is strictly related with the behavior at infinity of the density ρ. More precisely, if $N = 1$ or $N = 2$, then uniqueness of solutions holds if ρ merely belongs to L^∞ (see [6]). If instead $N \geq 3$ an additional requirement on ρ must be satisfied in order to get uniqueness, namely that $\rho(x)$ vanishes *slowly* as $|x|$ diverges, whereas nonuniqueness phenomena arise if the opposite behavior is satisfied by ρ (see [4,5,7,8,12,10]; see also [13] for $m = 1$).

If $\rho \equiv 1$ in (1.1), we get the following nonlocal version of the initial value problem for the porous medium equation:

$$\begin{cases}\partial_t u + (-\Delta)^{\frac{\sigma}{2}}\left[u^m\right] = 0 & x \in \mathbb{R}^N,\ t > 0\\ u = u_0 & x \in \mathbb{R}^N,\ t = 0.\end{cases} \tag{1.3}$$

This problem has been studied very recently in [14] where existence, uniqueness and properties of weak solutions to (1.3) have been established assuming $u_0 \in L^1(\mathbb{R}^N)$; the particular case $\sigma = 1$ has been addressed in [15]. Moreover, symmetrization techniques have been applied in [16,17]. A similar equation has also been studied in [18,19].

The study of problem (1.1) makes perfect sense, as it can be regarded both as a generalization of problem (1.3) and as nonlocal version of problem (1.2). Moreover, problem (1.1) arises in many physical situations such as diffusions in inhomogeneous media (see, *e.g.* [20–22]), and it is particularly interesting from a probabilistic point of view when $m = 1$ (see [23,24]).

Nonetheless, to the best of our knowledge, the analysis of problems like (1.1) is relatively new in the literature. Some results for nonlocal linear parabolic equation with a variable density have been established in [25], but not for problem considered in this paper. Recently, in [26], the special case $N = \sigma = 1$ has been studied, that is

$$\begin{cases}\rho\,\partial_t u + \left(-\dfrac{\partial^2}{\partial x^2}\right)^{\frac{1}{2}}\left[u^m\right] = 0 & x \in \mathbb{R},\ t > 0\\ u = u_0 & x \in \mathbb{R},\ t = 0.\end{cases} \tag{1.4}$$

In the light of results in [4,5,7,8], bounded initial data have been considered in [26]; existence and uniqueness of *very weak solutions* to problem (1.4) (namely solutions not having finite energy in the whole $\mathbb{R}^N$) have been proved in the class of bounded solutions not satisfying any extra conditions at infinity.

We point out that the arguments used in [26] are completely different from those in the present paper. In fact, as well as in [10,15,14], we deal here with *weak energy solutions* to problem (1.1) (see Definition 2.1), and consider initial data u_0 belonging to $L^1_\rho(\mathbb{R}^N)$. We emphasize also that our results differ from those in [15,14], where ρ is constant.

We outline next the structure and main contributions of this paper. In Section 2 after recalling the mathematical background about the fractional Laplacian, such as its realization through the harmonic extension, we give the precise notion of solution we will consider. In Section 3 we prove existence of weak energy solutions. The presence of the varying density ρ does not bring any additional technical difficulty at this stage of the work, and the proof of the main result of this section Theorem 3.1, goes along the same lines as the proof of the existence results in [15,14,26]; however we will sketch it for the search for completeness and further references.

In Sections 4 and 5 we deal with uniqueness and nonuniqueness of solutions. Concerning these issues, as expected, problem (1.1) turns out to share many aspects with its local counterpart, problem (1.2).

First, in Theorem 4.1 we establish uniqueness under the additional requirement that $\rho(x)$ vanishes *slowly* as $|x|$ diverges. As a byproduct, we show that total mass is conserved along the evolution; see Proposition 4.4.

The opposite case in which $\rho(x)$ decays *fast* as $|x| \to \infty$ is studied in Section 5. We first prove in Theorem 5.5 that in this case there exist solutions to (1.1) satisfying an extra condition at infinity (see (5.13)); the proof of these results makes use of a theorem shown in [27] and requires $N \geq 2$. As a consequence, in Corollaries 5.7 and 5.8 we easily obtain nonuniqueness of bounded solutions, if we do not specify their behavior at infinity. Instead, we shall prove that uniqueness is restored in the class of solutions satisfying a proper decay condition at infinity (see Theorem 5.9).

Finally, in Section 6 we study the particular situation in which $\sigma = 1$ in (1.1), that is:

$$\begin{cases}\rho\,\partial_t u + (-\Delta)^{\frac{1}{2}}\left[u^m\right] = 0 & x \in \mathbb{R}^N,\ t > 0\\ u = u_0 & x \in \mathbb{R}^N,\ t = 0.\end{cases} \tag{1.5}$$

In this case, it is possible to get rid of the boundedness assumption on the initial data and prove existence of solutions under the weaker hypothesis $u_0 \in L^1_\rho(\mathbb{R}^N)$ (see Theorem 6.5). A key tool, for this scope, is a smoothing estimate (see Theorem 6.2),