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Ergodicity for functional stochastic differential equations and applications[☆]

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ABSTRACT

In this paper, using the remote start or dissipative method, we investigate ergodicity for several kinds of functional stochastic equations including functional stochastic differential equations (SDEs) with variable delays, neutral functional SDEs, functional SDEs driven by jump processes, and semi-linear functional stochastic partial differential equations (SPDEs). Using the ergodicity derived, we then treat a couple of applications in stochastic approximation and optimization problems.

1. Introduction

The ergodicity of stochastic differential equations (SDEs) and stochastic partial differential equations (SPDEs), in which the state spaces are independent of the past history, has been studied extensively. There are several approaches to investigate such properties for finite or infinite-dimensional stochastic dynamic systems; see, for example, Mattingly et al. [1] and Rey-Bellet [2] using the Lyapunov function argument (Meyn and Tweedie [3]), Dong et al. [4] and Priola et al. [5] using Harris' theorem [1, Theorem 1.5], and Wang [6] and Zhang [7] using the coupling method. Further references on ergodicity of infinite-dimensional systems can also be found in the monograph [8] and the lecture notes [9].

More often than not, delays are unavoidable in a wide range of applications. In response to the great needs, there is an extensive literature on functional differential equations. We refer to Hale and Lunel [10] for functional ordinary differential equations, and Mohammed [11] for functional SDEs.

For functional SDEs, one of the classical methods for showing existence of an invariant measure is to exhibit an accumulation point of a sequence of Krylov–Bogoliubov measures [8, Theorem 3.1.1, p. 21] by using the tightness criterion of probability measures on the continuous function space [12, Theorem 8.5, p. 55]. To demonstrate the existence of an

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invariant measure, Es-Sarhir et al. [13] and Kinnally and Williams [14] considered functional SDEs with super-linear drift term and positivity constraints, respectively. Combining some exponential-type estimates, Bo and Yuan [15] investigated stochastic differential delay equations with jumps. It is also worthy to point out that Kolmogorov's tightness criterion [16, Problem 2.4.11, p. 64] plays an important role. To apply the Kolmogorov tightness criterion for the diffusion coefficients, we generally need to show the uniform p-th moment with $p > 2$ for the segment processes. Although the approach in Es-Sarhir et al. [13] and Kinnally and Williams [14] has been successfully used in investigation of existence of invariant measures for a wide range of functional SDEs, such methods are difficult to apply for *neutral* functional SDEs and functional SDEs driven by *jump processes*. Recently, Bo and Yuan [15] developed an approach from Rökner and Zhang [17] to cope with the case of functional SDEs driven by Poisson jump processes. However, their techniques are difficult to apply for neutral functional SDEs and cannot be generalized to infinite-dimensional SPDEs driven by jump processes because their approach is dimension dependent (see [15, p. 12]).

By Doob's theorem [8, Theorem 4.2.1, p. 43], Reiß et al. [18] obtained uniqueness of invariant measures for a class of linear functional SDEs with non-delayed diffusion coefficients, where the semigroup generated by the segment process is strong Feller. However, if the diffusion term depends on the past history, the semigroup generated by the segment process cannot be expected to be strong Feller; see Hairer et al. [19]. For such cases, Doob's theorem does not work for obtaining uniqueness of invariant measures. Recently, by an asymptotic coupling approach, Hairer et al. [19] addresses this problem for functional SDEs, where the diffusion coefficient is dependent on the segment, non-degenerate, uniformly bounded for the corresponding inverse, under some appropriate conditions, which may not ensure existence of an invariant measure [19, Remark 3.2, p. 237].

In this paper, under certain dissipative conditions, we present a unified approach (the remote start method or dissipative method) to establish the existence and uniqueness of invariant measures and the exponential ergodicity of the associated transition semigroups for several kinds of functional SDEs, which include functional SDEs with variable delays, neutral functional SDEs and functional SDEs driven by jump processes. Moreover, our method can also cover some infinite-dimensional semi-linear functional SPDEs driven by jump processes (such as cylindrical α-stable processes).

The rest of the paper is organized as follows. Using the remote start method, Section 2 presents the ergodicity for several kinds of functional SDEs including functional SDEs with variable delays, neutral functional SDEs and functional SDEs driven by Lévy processes, Section 3 focuses on the SPDE cases, and Section 4 provides a couple of examples of using the derived ergodicity to stochastic approximation and optimization problems. Before proceeding further, a few words about the notation are in order. Generic constants will be denoted by c; we use the shorthand notation $a \lesssim b$ to mean $a \le cb$. If the constant c depends on a parameter p, we shall also write $c(p)$ and $a \lesssim_p b$.

2. Ergodicity for functional SDEs

For each strictly positive integer n, let $\mathbb{R}^n$ be an n-dimensional Euclidean space endowed with the inner product $\langle u, v\rangle := \sum_{i=1}^n u^i v^i$ for $u, v \in \mathbb{R}^n$ and the Euclidean norm $|u| := \langle u, u\rangle^{1/2}$ for $u \in \mathbb{R}^n$. Let $\mathbb{R}^n \otimes \mathbb{R}^m$ denote the collection of all $n \times m$ matrices with real entries and A_{ij} means the entry of the ith row and the jth column. Given an $n \times m$ matrix A, $\|A\|_{HS} := \left(\sum_{i=1}^n \sum_{j=1}^m A_{ij}^2\right)^{1/2}$ denotes the Frobenius norm of A. For any two metric spaces $\mathbb{E}_1, \mathbb{E}_2$, let $C(\mathbb{E}_1; \mathbb{E}_2)$ denote the set of continuous functions from $\mathbb{E}_1$ into $\mathbb{E}_2$. Here, $\mathbb{E}_1$ will often be a closed interval $I \subset (-\infty, \infty)$, and $\mathbb{E}_2$ will often be $\mathbb{R}^n$ for various dimensions n. Fix $\tau \in (0, \infty)$, which will be referred to as the delay or time lag. Let $\mathscr{C} := C([-\tau, 0]; \mathbb{R}^n)$ equipped with the uniform norm $\|\zeta\|_\infty := \sup_{-\tau\le\theta\le 0} |\zeta(\theta)|$ for $\zeta \in \mathscr{C}$. For $X(\cdot) \in C([-\tau, \infty); \mathbb{R}^n)$ and $t \ge 0$, define the segment process $X_t \in \mathscr{C}$ by $X_t(\theta) := X(t+\theta), \theta \in [-\tau, 0]$. It should be pointed out that $X(t) \in \mathbb{R}^n$ is a point, while $X_t \in \mathscr{C}$ is a continuous function on the interval $[-\tau, 0]$ taking values in $\mathbb{R}^n$. By a filtered probability space, we mean a quadruple $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\ge0}, \mathbb{P})$, where $\mathscr{F}$ is a σ-algebra on the outcome space Ω, $\mathbb{P}$ is a probability measure on the measurable space $(\Omega, \mathscr{F})$, and $\{\mathscr{F}_t\}_{t\ge0}$ is a filtration of sub-σ-algebra of $\mathscr{F}$, where the usual conditions are satisfied, i.e., $(\Omega, \mathscr{F}, \mathbb{P})$ is a complete probability space, and $\mathscr{F}_0$ contains all $\mathbb{P}$-null sets of $\mathscr{F}$ and, for each $t \ge 0$, $\mathscr{F}_{t_+} := \cap_{s>t} \mathscr{F}_s = \mathscr{F}_t$. Let $\{W(t)\}_{t\ge0}$ be an m-dimensional Brownian motion defined on the filtered probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\ge0}, \mathbb{P})$. The notation $\mathcal{P}(\mathscr{C})$ denotes the family of all probability measures on $(\mathscr{C}, \mathscr{B}(\mathscr{C}))$, $\mathscr{B}_b(\mathscr{C})$ means the set of all bounded measurable functions $F : \mathscr{C} \to \mathbb{R}$ endowed with the norm $\|F\|_0 := \sup_{\phi\in\mathscr{C}} |F(\phi)|$, and $\mu(\cdot)$ stands for a probability measure on $[-\tau, 0]$. For any $F \in \mathscr{B}_b(\mathscr{C})$ and $\pi(\cdot) \in \mathcal{P}(\mathscr{C})$, let $\pi(F) := \int_{\mathscr{C}} F(\phi)\pi(\mathrm{d}\phi)$.

2.1. Ergodicity for functional SDEs driven by Wiener processes

Consider a functional SDE

$$\mathrm{d}X(t) = b(X_t)\mathrm{d}t + \sigma(X_t)\mathrm{d}W(t), \quad t > 0 \tag{2.1}$$

with the initial data $X_0 = \xi \in \mathscr{C}$, where $b : \mathscr{C} \to \mathbb{R}^n$, $\sigma : \mathscr{C} \to \mathbb{R}^n \otimes \mathbb{R}^m$ are measurable, locally bounded and continuous. Throughout this subsection, we assume that the initial value $\xi \in \mathscr{C}$ is independent of $\{W(t)\}_{t\ge0}$.