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Growth conditions for uniform asymptotic stability of damped oscillators

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ABSTRACT

The present paper is devoted to an investigation on the uniform asymptotic stability for the linear differential equation with a damping term,

$$x'' + h(t)x' + \omega^2 x = 0$$

and its generalization

$$(\phi_p(x'))' + h(t)\phi_p(x') + \omega^p \phi_p(x) = 0,$$

where $\omega > 0$ and $\phi_p(z) = |z|^{p-2}z$ with $p > 1$. Sufficient conditions are obtained for the equilibrium $(x, x') = (0, 0)$ to be uniformly asymptotically stable under the assumption that the damping coefficient $h(t)$ is integrally positive. The obtained condition for the damped linear differential equation is given by the form of a certain uniform growth condition on $h(t)$. Another representation which is equivalent to this uniform growth condition is also given. Our results assert that the equilibrium can be uniformly asymptotically stable even if $h(t)$ is unbounded. An example is attached to show this fact. In addition, easy-to-use conditions are given to guarantee that the uniform growth condition is satisfied. Moreover, a sufficient condition expressed by an infinite series is presented. The relation between the representation of an infinite series and the uniform growth condition is also clarified. Finally, our results are extended to be able to apply to the above-mentioned nonlinear differential equation.

1. Introduction

The equations considered in this paper are

$$x'' + h(t)x' + \omega^2 x = 0, \tag{1.1}$$

and its generalization, where the prime denotes d/dt, the coefficient $h(t)$ is continuous and nonnegative for $t \geq 0$, and the number ω is positive. Eq. (1.1) is called the *damped linear oscillator*. The only equilibrium of (1.1) is the origin $(x, x') = (0, 0)$. Our objective is to establish sufficient conditions on the damping coefficient $h(t)$ for the equilibrium to be uniformly asymptotically stable.

As is well known, the concept of uniform asymptotic stability is greatly different from the concept of (merely) asymptotic stability; that is,

$$\lim_{t\to\infty} x(t) = \lim_{t\to\infty} x'(t) = 0$$

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for every solution $x(t)$ of (1.1). To verify that the equilibrium is asymptotically stable, we have only to show that each solution of (1.1) and its derivative tend to zero as time t increases. It is not necessary to care about the asymptotic speed of each pair $(x(t), x'(t))$. On the other hand, we have to confirm that each pair $(x(t), x'(t))$ approaches the origin at the speed of the same level in order to prove that the equilibrium is uniformly asymptotically stable (see Section 2 about the strict definitions of asymptotic stability and uniform asymptotic stability). Here is the difficulty of the research of uniform asymptotic stability.

Uniform asymptotic stability concerning nonlinear differential equations has been investigated by many authors in relation to Lyapunov's direct method. Here, to explain an importance of the research of uniform asymptotic stability briefly, we consider the linear time-varying system given by

$$\mathbf{x}' = A(t)\mathbf{x} \tag{1.2}$$

with $A(t)$ being an $n \times n$ continuous matrix. System (1.2) has the zero solution, which is equivalent to the equilibrium of the corresponding n-order linear differential equation. Let $\|\mathbf{x}\|$ be the Euclidean norm of a vector $\mathbf{x} \in \mathbb{R}^n$. We denote the solution of (1.2) passing through a point $\mathbf{x}_0 \in \mathbb{R}^n$ at a time $t_0 \geq 0$ by $\mathbf{x}(t; t_0, \mathbf{x}_0)$. It is well-known that the zero solution of (1.2) is uniformly asymptotically stable if and only if it is exponentially asymptotically stable (or exponentially stable); namely, there exists a $\kappa > 0$ and, for any $\varepsilon > 0$, there exists a $\delta(\varepsilon) > 0$ such that $t_0 \geq 0$ and $\|\mathbf{x}_0\| < \delta(\varepsilon)$ imply $\|\mathbf{x}(t; t_0, \mathbf{x}_0)\| < \varepsilon \exp(-\kappa(t - t_0))$ for all $t \geq t_0$. Thanks to this characteristic of solutions of (1.2), we can obtain converse theorems on uniform asymptotic stability that guarantee the existence of a good Lyapunov function. The good Lyapunov function $V(t, \mathbf{x}) : [0, \infty) \times \mathbb{R}^n$ satisfies

(i) $a(\|\mathbf{x}\|) \leq V(t, \mathbf{x}) \leq b(\|\mathbf{x}\|)$,
(ii) $\dot{V}_{(1.2)}(t, \mathbf{x}) \leq -c(\|\mathbf{x}\|)$ or $\dot{V}_{(1.2)}(t, \mathbf{x}) \leq -dV(t, \mathbf{x})$,
(iii) $|V(t, \mathbf{x}_1) - V(t, \mathbf{x}_2)| \leq f(t)\|\mathbf{x}_1 - \mathbf{x}_2\|$,

where $a(\cdot)$, $b(\cdot)$ and $c(\cdot)$ are continuous increasing and positive definite functions, d is a positive constant and $f(t)$ is a positive suitable function. In general, however, (merely) asymptotic stability of the zero solution of a time-varying system does not ensure the existence of any good Lyapunov function (see [1, Example 2]). This point is a big difference between uniform asymptotic stability and asymptotic stability. A function satisfying the above properties (i) and (ii) is often called a *strict* Lyapunov function in control theory (for example, see [2, pp. 101–103]). We can solve perturbation problems by utilizing such a good Lyapunov function. For example, if the zero solution of (1.2) is uniformly asymptotically stable and if $\mathbf{g}(t, \mathbf{x})$ and $\lambda(t)$ satisfy that $\|\mathbf{g}(t, \mathbf{x})\| \leq \lambda(t)\|\mathbf{x}\|$ for $t \geq 0$ and $\mathbf{x} \in \mathbb{R}^n$, where

$$\int_0^\infty \lambda(t)dt < \infty,$$

then the zero solution of the quasi-linear system

$$\mathbf{x}' = A(t)\mathbf{x} + \mathbf{g}(t, \mathbf{x})$$

is also uniformly asymptotically stable. However, even if the zero solution of (1.2) is (merely) asymptotically stable, the zero solution of the quasi-linear system is not always asymptotically stable. Perron [3] has clarified this fact by considerably complicated analysis. For example, the reader is referred to the classical books [4, pp. 42–43], [5, pp. 169–170], [6, p. 71]. It is also known that the zero solution of (1.2) is uniformly asymptotically stable if and only if it is totally stable which is closely related to robustness. For the definition of total stability, see [2, p. 45] and [7, pp. 118–119].

Let $X(t)$ be a fundamental matrix for a general n-dimensional linear system satisfying $X(0) = E$; E denotes the unit matrix. We define the norm of $X(t)$ to be

$$\|X(t)\| = \sup_{\|\mathbf{x}\|=1} \|X(t)\mathbf{x}\|.$$

It is well-known that the zero solution of (1.2) is asymptotically stable if and only if

$$\|X(t)\| \to 0 \quad \text{as } t \to \infty$$

and that the zero solution of (1.2) is uniformly asymptotically stable if and only if there exist positive constants K and κ such that

$$\|X(t)X^{-1}(s)\| \leq K \exp(-\kappa(t - s)) \quad \text{for } 0 \leq s \leq t < \infty$$

(for the proof, see the books [6, p. 54] or [8, p. 84]). If we can get a concrete expression of a fundamental matrix, we may be able to judge whether the zero solution is uniformly asymptotically stable (or asymptotically stable) by using the above-mentioned criterion. Unfortunately, however, we are almost unable to find a fundamental matrix. Therefore, these criteria are not useful for practical use though they are sharp.

Before going into the main theme, let us look at the results concerning the asymptotic stability. Many papers have been written to find out sufficient conditions and necessary conditions for the zero solution (or the equilibrium) to be asymptotically stable without using the information on a fundamental matrix (for example, see [9–17]). Historical progress of this research is briefly summarized in Sugie [18, Section 1]. Here, we will describe some results of not having written to the summary.