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Counterexample of loss of regularity for fractional order evolution equations with both degenerating and oscillating coefficients

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ABSTRACT

For weak evolution models of fractional order with singularity near the origin, the joint influence from the principal σ-Laplacian operator, degenerating part and oscillating part is of prime concern in the discussion of regularity behavior of the solutions. We apply the techniques from the micro-local analysis to explore the upper bound of loss of regularity. Furthermore, in order to demonstrate the optimality of the estimates, a delicate counterexample with periodic coefficients will be constructed to show the lower bound of loss of regularity by the application of harmonic analysis and instability arguments. This optimality discussion develops the theory in Cicognani and Colombini (2006), Cicognani et al. (2008), Lu and Reissig (2009) and Lu and Reissig (2009) by combining both oscillation and degeneracy of the coefficients.

1. Introduction

Pseudodifferential operators, especially fractional order operators (also called Riesz fractional derivatives) are very important mathematical models which describe plenty of anomalous dynamic behaviors in our daily life, such as charge carrier transport in amorphous semiconductors, nuclear magnetic resonance diffusometry in percolative and porous media, transport on fractal geometries, diffusion of a scalar tracer in an array of convection rolls, dynamics of a bead in a polymeric network, transport in viscoelastic materials, etc.

Generally speaking, loss of regularity, or loss of derivatives, is a phenomenon in which the solution loses certain regularity in the sense of Sobolev spaces compared with initial Cauchy data. This is a very important aspect of the well-posedness research [1]. First we recall the regularity behavior of the evolutional operator on $[0, T] \times \mathbb{R}$, $\mathscr{L} = \partial_t^2 - \lambda^2(t)\partial_x^2$, where $\lambda(t)$ is the measure function of degeneracy, which is defined as follows.

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- Let $\Lambda(t) \triangleq \int_0^t \lambda(\tau)d\tau$, a measure function of degeneracy $\lambda(t) \in C^2$ is a positive function satisfying:

$$\lambda(0) = 0, \qquad \lambda'(t) > 0, \qquad \frac{\lambda'(t)}{\lambda(t)} \sim \frac{\lambda(t)}{\Lambda(t)}, \qquad |\lambda''(t)| \lesssim \lambda(t)\left(\frac{\lambda(t)}{\Lambda(t)}\right)^2.$$

It is worth noticing that the above model precisely generalizes the weakly hyperbolic operators with infinitely/finitely degenerating coefficients [2,3]. By applying the diagonalization techniques introduced in [4] while considering the propagation of mild singularities for semi-linear weakly hyperbolic equations, we know the fact that there exists no loss of regularity for this kind of operator.

In order to consider the impact of oscillation on the regularity behavior from the principal elliptic operator, we introduced in [5,6] a brand-new weakly hyperbolic operator with both oscillating and degenerating coefficients on $[0, T] \times \mathbb{R}$: $\mathscr{L} = \partial_t^2 - \lambda^2(t)b^2(t)\partial_x^2$, where $\lambda(t)$ is the measure function of degeneracy, $b(t) \in C^2(0, T]$ describes the oscillation of the principal elliptic operator near the origin 0. In physics, $\lambda(t)$ usually describes the degenerating part of the density and $b(t)$ often describes the oscillating part of the density. More precisely,

- $b_0 \triangleq \inf_{t\in(0,T]} b(t) \le b(t) \le b_1 \triangleq \sup_{t\in(0,T]} b(t),\ b_0, b_1 > 0$;
- $|b^{(k)}(t)| \le C\left(\frac{\lambda(t)}{\Lambda(t)}\nu(t)\right)^k$, $C > 0, k = 1, 2$, where $\nu(t)$ is a measure function of oscillation, which is defined as a continuous and decreasing positive function on a finite time interval.

By two steps of diagonalization procedure, we have an insightful understanding of the impact from the oscillating coefficients. Detailed description of both the loss of regularity and difference of regularity of the initial Cauchy data is given in [6]. In the previous literature [4,6–11], one mainly discussed the wave equation with lower order terms. As a matter of fact, the finite propagation speed holds for this kind of operator. In reality, most of the operators have infinite propagation speed, such as heat equation $u_t - \Delta u = 0$, Petrowsky equation $u_{tt} + \Delta^2 u = 0$, etc. In this manuscript, the weak evolution operator of fractional order with oscillating and degenerating coefficients, which has infinite propagation speed, is of prime concern:

$$\mathscr{L} = \partial_t^2 + A_0(t, \sqrt{-\Delta}), \tag{1}$$

where

$$A_0(t, \sqrt{-\Delta}) \triangleq \lambda^2(t)b^2(t)(-\Delta)^\sigma,$$

with $\sigma > 1$ and $(-\Delta)^\sigma$ defined on the torus $\mathbb{T}^N$. In this model, we call $A_0(t, \sqrt{-\Delta})$ the principal part in the sense of Petrowsky. One typical example of the coefficients on the principal part is $b(t) = 2 + \sin\big((\log(1/t))^\kappa\big)$, $\kappa \in (1, \infty)$, which satisfies the assumptions with $\nu(t) = (\log(1/t))^{\kappa-1}$. Up until now, there is still no complete conclusion about the Levi-condition with oscillation [8]. However, the theory of pseudodifferential operators assures the existence and uniqueness of the solution for the Cauchy problem of (1). As an important application of this model, actually, through Nirenberg's transformation $v = 1 - \exp(-u)$, the problem of the semi-linear Cauchy problem $u_{tt} - a^2(t)\Delta u = u_t^2 - a^2(t)|\nabla u|^2$ can be turned into the linear problem $v_{tt} - a^2(t)\Delta v = 0$. For more discussion in this respect please refer to [12]. In the following, we explore carefully the joint influence upon the regularity behavior of (1) from both oscillation and degeneracy of the principal elliptic operator $A_0(t, \sqrt{-\Delta})$.

Under the above assumptions of $\lambda(t)$ and $b(t)$, one has the following regularity statement.

Theorem 1.1. *Let us consider the Cauchy problem of model (1) on* $[0, T] \times \mathbb{T}^N$,

$$\mathscr{L}u = 0, \qquad u(0, x) = u_0(x), \qquad u_t(0, x) = u_1(x). \tag{2}$$

If the initial Cauchy data satisfy

$$u_0 \in H^s(\mathbb{T}^N), \qquad u_1 \in \frac{1}{\Lambda^{-1}\left(\frac{2^{P_1}}{(\sqrt{1-\Delta})^\sigma}\right)} H^s(\mathbb{T}^N),$$

where $P_1 \in \mathbb{N}_+$ is a fixed constant and the Sobolev index s is sufficiently large, then there exists a unique solution u in the following function spaces:

$$u \in C\left([0, T], \exp\left(C_\alpha \nu\left(\left(\frac{\Lambda}{\nu}\right)^{-1}\left(\frac{2^{P_2}}{(\sqrt{1-\Delta})^\sigma}\right)\right)\right) H^s(\mathbb{T}^N)\right),$$

$$u_t \in C\left([0, T], \exp\left(C_\alpha \nu\left(\left(\frac{\Lambda}{\nu}\right)^{-1}\left(\frac{2^{P_2}}{(\sqrt{1-\Delta})^\sigma}\right)\right)\right) H^{s-\sigma}(\mathbb{T}^N)\right);$$

where $C_\alpha \in \mathbb{R}_+$ and $P_2 \in \mathbb{N}_+$ are fixed constants. In this theorem, Λ^{-1} and $\left(\frac{\Lambda}{\nu}\right)^{-1}$ denote respectively the corresponding inverse functions. In fact, according to the monotonicity of $\lambda(t)$ and $\nu(t)$, both inverse functions are well-defined.