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Existence and some properties of solutions for degenerate elliptic equations with exponent variable

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ABSTRACT

In this paper, we study degenerate elliptic equations with variable exponents when a perturbation term satisfies the Ambrosetti–Rabinowitz condition and does not satisfy the Ambrosetti–Rabinowitz condition. For the first case, we employ the standard Mountain Pass theorem to give the existence of solutions. For the second case, we use Browder's theorem for monotone operators to show the unique existence of a solution when the perturbation term is decreasing with respect to a function variable. A priori bound and nonnegativeness of solutions are also given. We emphasize that the log-Hölder continuous condition is not required.

1. Introduction

We are concerned with the existence and uniqueness of solutions for the degenerate $p(x)$-Laplacian with Dirichlet boundary condition as follows;

$$\begin{cases} -\operatorname{div}\left(w(x)|\nabla u|^{p(x)-2}\nabla u\right) = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with Lipschitz boundary $\partial\Omega$, the variable exponent $p : \overline{\Omega} \to (1,\infty)$ a continuous function and w a measurable positive a.e. finite function in Ω and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies a Carathéodory condition.

Recently, many mathematicians have intensively studied $p(x)$-Laplacian [1–11] which is dealing with nonsmooth growth. $p(x)$-Laplacian can be found in the areas, electro-rheological fluids [12], the thermistor problem [13], or the problem of image recovery [2]. When w is not bounded and/or not separated from zero, w is called degenerate (or singular). A degenerated second order linear differential operator was basically due to Murthy and G. Stampacchia [14] and higher order linear degenerated elliptic operators were extended in the 80s and quasilinear elliptic equations including p-Laplacian were developed in the 90s (see [15]). Degenerate phenomena appear in the area of oceanography, turbulent fluid flows, induction heating, and electrochemical problems.

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The goal of this paper is to get the various existence and uniqueness results of solutions for (1.1) under suitable conditions of w and f. Fortunately, Kim, Wang and Zhang [16] have shown good properties of a function space, the so-called weighted variable exponent Lebesgue–Sobolev spaces (see Section 2). We shall employ the standard Mountain Pass theorem when f satisfies the Ambrosetti–Rabinowitz condition ((AR)-condition) [17] as usual. It is worth noting that the growth condition of the paper is a little different from that of [18] and we do not assume that the exponent $p(x)$ is log-Hölder continuous, i.e., there is a constant C such that

$$|p(x) - p(y)| \leq \frac{C}{-\log|x-y|} \tag{1.2}$$

for every $x, y \in \Omega$ with $|x-y| \leq 1/2$. Besides, we shall show the unique existence of solutions for (1.1) using Browder's theorem for monotone operators in the reflexive Banach spaces when f is nonincreasing with respect to the u-variable. Finally, we shall show that a weak solution of (1.1) is bounded under a little restricted conditions of w and f adopting the De Giorgi iteration and localization method. Using this fact and the cut-off method, we shall prove the nonnegativeness of the solution for (1.1).

One of the novelties of this paper is that we do not assume that $p(x)$ is log-Hölder continuous. When $p(x)$ is log-Hölder continuous, an easier proof for Lemma 4.1 can be given. The other one is to give a suitable condition of w to guarantee a continuous imbedding which completes the De Giorgi iteration argument and obtains an a-priori bound of a weak solution for (1.1). Unlike the constant variable exponent case, the case of $p_s(x) \geq N$ (see, Section 2 for the definition) is not obvious. So we need more arguments to complete in some steps. Finally, we try to make the paper self-contained, that is, we give the detailed proof to understand easily.

This paper is organized as follows. In Section 2, we define the weighted variable exponent Lebesgue–Sobolev spaces and list properties of that space. In Section 3, we show the existence of a solution for (1.1) in two cases; with the (AR)-condition, and without the (AR)-condition using the Mountain Pass theorem and Browder's theorem, respectively. In Section 4, employing the De Giorgi iteration and localization method, we obtain an a priori bound of a weak solution for (1.1). Finally, we prove that a solution of (1.1) is nontrivial and nonnegative in Section 5.

2. Abstract framework and preliminary results

In this section, we define the weighted variable exponent Lebesgue–Sobolev spaces and list properties of that space. Since the variable exponent Lebesgue–Sobolev spaces $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$ were thoroughly studied in [4–6,9], we shall only review the weighted variable exponent Lebesgue–Sobolev spaces $L^{p(x)}(w,\Omega)$ and $W^{1,p(x)}(w,\Omega)$, which were studied in [16].

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with Lipschitz boundary $\partial\Omega$ and $w(x)$ be a weight function on Ω. Set

$$C_+(\overline{\Omega}) = \left\{ p \in C(\overline{\Omega}) : \min_{x\in\overline{\Omega}} p(x) > 1 \right\}.$$

Let $p \in C_+(\overline{\Omega})$ and denote

$$p^- := \min_{x\in\overline{\Omega}} p(x), \qquad p^+ := \max_{x\in\overline{\Omega}} p(x).$$

For any $p \in C_+(\overline{\Omega})$, we define the variable exponent Lebesgue space by

$$L^{p(x)}(w,\Omega) = \left\{ u : \Omega \to \mathbb{R} \text{ is measurable}, \int_\Omega w(x)|u(x)|^{p(x)}dx < \infty \right\}.$$

Then $L^{p(x)}(w,\Omega)$ endowed with the norm

$$|u|_{L^{p(x)}(w,\Omega)} = \inf\left\{ \lambda > 0 : \int_\Omega w(x)\left|\frac{u(x)}{\lambda}\right|^{p(x)} dx \leq 1 \right\},$$

becomes a normed space. When $w(x) \equiv 1$ we have $L^{p(x)}(w,\Omega) \equiv L^{p(x)}(\Omega)$ and we use the notation $|u|_{L^{p(x)}(\Omega)}$ instead of $|u|_{L^{p(x)}(w,\Omega)}$.

The following Hölder type inequality is useful for the next sections.

Proposition 2.1 ([5,9]). *The space $L^{p(x)}(\Omega)$ is a separable, uniform convex Banach space, and its conjugate space is $L^{p'(x)}(\Omega)$ where $1/p(x) + 1/p'(x) = 1$. For any $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$ we have*

$$\left|\int_\Omega uv\,dx\right| \leq \left(\frac{1}{p^-} + \frac{1}{(p')^-}\right)|u|_{L^{p(x)}(\Omega)}|v|_{L^{p'(x)}(\Omega)} \leq 2|u|_{L^{p(x)}(\Omega)}|v|_{L^{p'(x)}(\Omega)}.$$

The modular of the space $L^{p(x)}(w,\Omega)$, which is the mapping $\rho : L^{p(x)}(w,\Omega) \to \mathbb{R}$ is defined by

$$\rho(u) = \int_\Omega w(x)|u(x)|^{p(x)}dx, \quad \forall u \in L^{p(x)}(w,\Omega).$$