Nonlinear Analysis 98 (2014) 226–236

Contents lists available at ScienceDirect

Nonlinear Analysis

journal homepage: www.elsevier.com/locate/na

Global solution to Maxwell–Dirac equations in $1+1$ dimensions

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ARTICLE INFO

Article history:
Received 18 April 2013
Accepted 20 December 2013
Communicated by Enzo Mitidieri

Keywords:
Maxwell–Dirac equations
Global solution
Charge mass
Arzela–Ascoli

ABSTRACT

The Maxwell–Dirac equations with nonzero charge mass in one space dimension are studied under the Lorenz gauge condition. The global existence and uniqueness of the solution in $(L^2_{loc}(R^2_+))^2 \times (L^\infty_{loc}(R^2_+))^2$ for an initial value problem of Maxwell–Dirac equations are proved.

1. Introduction

We are concerned with the Maxwell–Dirac equations in one space dimension

$$\begin{cases} (i\gamma^\mu D_\mu + mI)\Psi = 0, \\ \partial_\mu F^{\mu\nu} = J^\nu, \end{cases} \tag{1.1}$$

under the Lorenz gauge condition

$$\partial_t A_0 - \partial_x A_1 = 0, \tag{1.2}$$

with

$$\begin{cases} \Psi_j(x,0) = \psi_j(x) \quad (j=1,2), \\ A_\nu(x,0) = a^0_\nu(x) \quad (\nu = 0,1), \\ \partial_t A_\nu(x,0) = a^1_\nu(x) \quad (\nu=0,1), \end{cases} \tag{1.3}$$

which satisfy the constraint

$$a^1_0 - \partial_x a^0_1 = 0,$$

and $\psi_j \in L^2_{loc}(R^1)$, $a^0_\nu \in L^\infty_{loc}(R^1)$, $a^1_\nu \in L^1_{loc}(R^1)$. Here $D_\mu = \partial_\mu - iA_\mu$ is the covariant derivative and $F^{\mu\nu} = \partial_\nu A_\mu - \partial_\mu A_\nu$ is the curvature associated with the gauge field $A_\mu \in R$. Ψ denotes a two-spinor field defined on R^{1+1}, $J^\nu = \Psi^\dagger \gamma^0 \gamma^\nu \Psi$ is a current density and $\Psi^\dagger = (\overline{\Psi_1}, \overline{\Psi_2})$ denotes the complex conjugate transpose of Ψ. $\partial_0 = \partial_t$, $\partial_1 = \partial_x$.

The Dirac gamma matrices are of the following:

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \gamma^1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

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http://dx.doi.org/10.1016/j.na.2013.12.014

The system (1.1) can be rewritten as follows,

$$\begin{cases} \partial_t \Psi_1 + \partial_x \Psi_2 = im\Psi_1 + iA_0\Psi_1 + iA_1\Psi_2, \\ \partial_t \Psi_2 + \partial_x \Psi_1 = -im\Psi_2 + iA_0\Psi_2 + iA_1\Psi_1, \\ \Box A_0 = |\Psi_1|^2 + |\Psi_2|^2, \\ \Box A_1 = -(\overline{\Psi_2}\Psi_1 + \overline{\Psi_1}\Psi_2), \\ \partial_t A_0 - \partial_x A_1 = 0. \end{cases} \tag{1.4}$$

The global well-posedness of a classical solution for the Maxwell–Dirac system in R^{1+1} has been established in [2]. Many works are devoted to study the existence and uniqueness of the solution in different functional spaces since then, see for instance [1–11] and the references therein. Recently in [10] Huh proved the global well-posedness of the strong solutions for the Maxwell–Dirac system (1.1) in R^{1+1}, where he assumed that the mass of charge is zero, that is, $m = 0$ in (1.1), and the solutions can be obtained by the explicit formula. For $m > 0$, as far as we know, there is no explicit formula for solutions. In this paper we consider more general case than that in [10], that is, for $m \geq 0$ we will find the $(\Psi, A) \in (L^2_{loc}(R^2_+))^2 \times (L^\infty_{loc}(R^2_+))^2$ which solves Eq. (1.4) with initial data (1.3) in the following sense and prove its uniqueness.

Definition 1.1. (Ψ, A) with $\Psi \in L^2_{loc}(R^2_+) \times L^2_{loc}(R^2_+)$ and $A \in L^\infty_{loc}(R^2_+) \times L^\infty_{loc}(R^2_+)$ is called a weak solution to (1.4) with the initial data (1.3) provided that

$$\iint_{t>0} \big(\Psi_1(\phi_{1t} + \phi_{1x} + im\phi_1) + i(A_0\Psi_1 + A_1\Psi_2)\phi_1\big)dxdt = \int_{-\infty}^{\infty} \psi_1\phi_1(x, 0)dx,$$
$$\iint_{t>0} \big(\Psi_2(\phi_{2t} + \phi_{2x} + im\phi_2) + i(A_0\Psi_2 + A_1\Psi_1)\phi_2\big)dxdt = \int_{-\infty}^{\infty} \psi_2\phi_2(x, 0)dx,$$

and

$$\iint_{t>0} \big(A_0\Box\phi_3 - (|\Psi_1|^2 + |\Psi_2|^2)\phi_3\big)dxdt = \int_{-\infty}^{\infty} \big(-a_0^0\phi_{3t}(x, 0) + a_0^1\phi_3(x, 0)\big)dx,$$
$$\iint_{t>0} \big(A_1\Box\phi_4 + (\overline{\Psi_2}\Psi_1 + \overline{\Psi_1}\Psi_2)\phi_4\big)dxdt = \int_{-\infty}^{\infty} \big(-a_1^0\phi_{4t}(x, 0) + a_1^1\phi_4(x, 0)\big)dx,$$

and

$$\iint_{t>0} (A_0\phi_{5t} + A_1\phi_{5x})dxdt = \int_{-\infty}^{\infty} a_0^0\phi_5(x, 0)dx,$$

for any $\phi_k \in C_c^\infty(\overline{R^2_+})$, $k = 1, 2, 3, 4, 5$. Here and in the sequel, by $\phi \in C_c^\infty(\overline{R^2_+})$ we denote that $\phi \in C^\infty(\overline{R^2_+})$ with bounded support in $\overline{R^2_+}$, where $R^2_+ = \{(x, t)|t > 0, x \in R^1\}$.

We remark that a solution in the sense considered here is also a distributional solution in the standard sense.

In the sequel, we use the notations $(L^2_{loc})^2 = L^2_{loc} \times L^2_{loc}$ and $(L^\infty_{loc})^2 = L^\infty_{loc} \times L^\infty_{loc}$ etc. for simplification. Now the main results are presented as follows.

Theorem 1.1. *For the initial data $\psi_j \in L^2(R^1)$ $(j = 1, 2)$ and $a_\mu = (a_\mu^0, a_\mu^1) \in L^\infty(R^1) \times L^1(R^1)$ $(\mu = 0, 1)$, there exists a global weak solution (Ψ, A) to (1.4) with (1.3), which satisfy*

$$\Psi = (\Psi_1, \Psi_2) \in C([0, \infty); L^2(R^1) \times L^2(R^1)),$$

and

$$A = (A_0, A_1) \in L^\infty(R^1 \times [0, T]) \times L^\infty(R^1 \times [0, T])$$

for any $T > 0$.

Theorem 1.2. *For $\psi_j \in L^2_{loc}(R^1)$ and $a_\mu = (a_\mu^0, a_\mu^1) \in L^\infty_{loc}(R^1) \times L^1_{loc}(R^1)$ $(j = 1, 2, \mu = 0, 1)$, there exists a unique global weak solution (Ψ, A) to (1.4) with (1.3), which satisfy*

$$\Psi_j \in L^2_{loc}(R^2_+), \qquad A_\mu \in L^\infty_{loc}(R^2_+),$$

for $j = 1, 2$ and $\mu = 0, 1$. Here and in the sequel, we denote $R^2_+ = \{(x, t)|t > 0, x \in R^1\}$.

The remaining part of the paper is organized as follows. In Section 2, we rewrite (1.1) and (1.2) in the equivalent form as (2.1)–(2.5) and present two theorems, Theorems 2.1 and 2.2, which are equivalent to main results, Theorems 1.1 and 1.2. In Section 3, based on Chadam's result on the global H^1 strong solution for (1.1)–(1.3) with initial data $(\psi, a_\nu^0, a_\nu^1) \in (H^1(R^1))^2 \times H^1(R^1) \times L^2(R^1)$, we establish the key estimates in Lemmas 3.1 and 3.3 for classical solutions to (2.1)–(2.5), and get the uniform boundness on the solutions. In Section 4, the precompactness of the approximate solutions $\{(u^n, v^n, A^n_\pm)\}_{n=1}^\infty$ is proved via the estimates given in Lemmas 3.1 and 3.3. Then we can get a convergent subsequence of $\{(u^n, v^n, A^n_\pm)\}_{n=1}^\infty$,