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Nonlinear gradient estimates for parabolic problems with irregular obstacles

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ABSTRACT

We establish the natural Calderón–Zygmund theory for solutions to parabolic variational inequalities satisfying an irregular obstacle constraint and involving degenerate/singular operators in divergence form of general type, and proving that the (spatial) gradient of solutions is as integrable as both the (spatial) gradient of the obstacles and the inhomogeneous terms, under the assumption that the involved nonlinearities have small a BMO semi-norm in the spatial variables while they are allowed to be merely measurable in the time variable.

1. Introduction

There has been a rapid scientific development in the theory of parabolic variational inequalities and their applications. These variational inequalities are systematically used in the theory of many practical problems related to an equilibrium state in Physics, Biology, Engineering, Economics, etc., see [1–5].

The problems of minimizing the energy functional with certain constraints are expressed as variational inequalities, and so the obstacle problems are naturally related with variational inequalities. The model of parabolic problems with constraints are used in many applied fields such as problems describing diffusion with semipermeable membrane, growth of tumor in Mathematical biology, the valuation model of the American option and so on.

In this paper we are concerned with regularity properties of solutions to parabolic obstacle problems. In particular, we wish to investigate the regularity estimates of solutions to parabolic variational inequalities with time dependent irregular obstacles and verify the natural Calderón–Zygmund theory when the associated nonlinearity is allowed to be merely measurable in the time variable.

There have already been many regularity results for the irregular obstacle problems. W. Ziemer treated parabolic obstacle problems with a mild condition on time independent obstacles in [6] where it was proved that the solution is continuous although the obstacle is allowed to be discontinuous. In [7] H. Choe considered parabolic obstacle problems with quadratic growth, to prove the Hölder continuity of the bounded weak solution and the Hölder continuity of its gradient in the interior. In [8] P. Lindqvist and M. Parviainen considered the obstacle problems for the evolutionary p-Laplacian equations when the obstacle is discontinuous and does not have regularity in the time variable, to show that some uniqueness and convergence

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results are no longer valid for this very irregular obstacle problem. In [9] V. Bögelein and C. Scheven established the self-improving property of integrability for the parabolic obstacle problems when the involved nonlinearity is allowed to be degenerate respectively singular, and satisfies merely the growth and monotonicity assumptions which are needed for the existence of a solution. In [10] V. Bögelein, F. Duzaar and G. Mingione developed the natural local Calderón–Zygmund theory for solutions to elliptic and parabolic variational problems with irregular obstacles. In [11] C. Scheven introduced the concept of localizable solutions to nonlinear parabolic obstacle problems and constructed the Calderón–Zygmund estimates using a truncation technique.

Our work is a natural follow-up of the very fine work [10] where a regular nonlinearity of class C^1 is considered. Here we treat a more general class of nonlinearities by allowing the involved nonlinearity to be merely measurable. Thereby we do not impose any regularity assumption on the nonlinearity in the time variable for a complete Calderón–Zygmund theory.

In the context of elliptic obstacle problems of general type, we would like to mention an earlier work [12] where the global Calderón–Zygmund theory was achieved via the maximal function operator. Knowing that the problem under consideration scales differently in space and time, no maximal function operator is applicable to the problem. Therefore we shall rather employ the so-called maximal function-free technique which was introduced in the very influential papers [13,14] for nonlinear Calderón–Zygmund estimates for parabolic systems of p-Laplacian type.

The organization of the paper will be as follows. In the next section we will present some preliminaries regarding irregular parabolic obstacle problems, describe the main regularity requirement on the nonlinearity of coefficients and state the main results. In Section 3 we study a local estimate of solutions to the variational inequality by approximation based on weak compactness methods for nonlinear parabolic equations. The a priori estimate for the parabolic problems with an irregular obstacle is the subject of Section 4. Section 5 will be devoted to an approximation procedure to remove the a priori assumption for the sake of completeness.

Finally, we should mention that in this paper we shall provide a detailed proof only for the local regularity. However, the global results can be also achieved, since with the local one on hand it relies essentially on the techniques employed in the previous works [12,15] in which elliptic problems were studied for the global gradient estimates on non-smooth domains.

2. Results

Let p be a fixed number with $\frac{2n}{n+2} < p < \infty$ and assume that a given Carathéodory function,

$$a = a(\xi, x, t) = a(\xi, z) : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n,$$

satisfies the following basic structural conditions:

$$\begin{cases} |a(\xi, x, t)| + |\xi||D_\xi a(\xi, x, t)| \le \Lambda|\xi|^{p-1}, \\ D_\xi a(\xi, x, t)\eta \cdot \eta \ge \mu|\xi|^{p-2}|\eta|^2, \end{cases} \tag{2.1}$$

for every $\xi, \eta \in \mathbb{R}^n$, for almost every $(x, t) \in \mathbb{R}^n \times \mathbb{R}$, and for some constants $0 < \mu \le 1 \le \Lambda$. We clearly point out that the conditions (2.1) imply the following standard monotonicity conditions:

$$\begin{cases} (a(\xi, x, t) - a(\eta, x, t)) \cdot (\xi - \eta) \ge \gamma|\xi - \eta|^p, \quad \text{if } p \ge 2, \\ (a(\xi, x, t) - a(\eta, x, t)) \cdot (\xi - \eta) \ge \gamma|\xi - \eta|^2(|\xi| + |\eta|)^{p-2}, \quad \text{if } 1 < p < 2, \end{cases} \tag{2.2}$$

where γ is a positive constant depending only on n, μ and p.

Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$, $n \ge 2$, and write $\Omega_T = \Omega \times (a, a+T)$ for some constants $a \in \mathbb{R}$ and $T > 0$. We then consider a function ψ as the time dependent obstacle with

$$\psi \in L^p(a, a+T; W^{1,p}(\Omega)), \quad \psi_t \in L^{\frac{p}{p-1}}(\Omega_T) \quad \text{and} \quad \psi \le 0 \text{ a.e. on } \partial\Omega \times (a, a+T), \tag{2.3}$$

and a measurable function F as the inhomogeneity with

$$F \in L^p(\Omega_T, \mathbb{R}^n). \tag{2.4}$$

For the sake of simplicity, we take zero initial value, as we assume

$$u(\cdot, a) = 0 \quad \text{and} \quad 0 \ge \psi(\cdot, a). \tag{2.5}$$

A solution under consideration is a function $u = u(x, t)$ lying in the convex admissible set

$$\mathcal{A} = \{v \in C^0([a, a+T]; L^2(\Omega)) \cap L^p(a, a+T; W_0^{1,p}(\Omega)) : v(\cdot, a) = 0, v \ge \psi\} \tag{2.6}$$

and satisfying the weak parabolic variational inequality

$$\int_a^{a+T} \langle v_t, v - u\rangle dt + \int_{\Omega_T} a(Du, x, t) \cdot D(v - u)dxdt \ge \int_{\Omega_T} |F|^{p-2}F \cdot D(v - u)dxdt, \tag{2.7}$$

for all testing functions $v \in \mathcal{A}'$, where

$$\mathcal{A}' = \{v \in \mathcal{A} : v_t \in L^{\frac{p}{p-1}}(a, a+T; W^{-1,\frac{p}{p-1}}(\Omega))\}, \tag{2.8}$$

and $\langle \cdot, \cdot\rangle$ denotes the duality pairing between $W^{-1,\frac{p}{p-1}}$ and $W_0^{1,p}$.