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On the boundedness of the multilinear fractional integral operators

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ABSTRACT

We derive necessary and sufficient conditions for a weight v governing the weighted inequality

$$\left(\int_{\mathbb{R}^n} \left|\mathcal{I}_\alpha(\vec{f})(x)\right|^q v(x)dx\right)^{1/q} \le C\prod_{i=1}^m\left(\int_{\mathbb{R}^n} |f_i(x)|^{p_i}dx\right)^{1/p_i},$$

where $\mathcal{I}_\alpha$ is the multilinear fractional integral operator. The derived condition is of D. Adams type and any additional assumption on the weight v is not assumed.

1. Introduction

It is well known that the fractional integral operator I_α $(0 < \alpha < n)$, defined by

$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}\,dy, \quad x \in \mathbb{R}^n$$

for appropriate functions f, plays a fundamental role in harmonic analysis, mainly in the theory of Sobolev embeddings.

The present paper is devoted to investigating the multilinear variant of the fractional integral operators on weighted L_p-spaces. The main purpose is to derive necessary and sufficient conditions for a weight v governing the weighted inequality

$$\left(\int_{\mathbb{R}^n} |\mathcal{I}_\alpha(\vec{f})(x)|^q v(x)dx\right)^{1/q} \le C\prod_{i=1}^m\left(\int_{\mathbb{R}^n} |f_i(x)|^{p_i}dx\right)^{1/p_i}, \tag{1}$$

where $\mathcal{I}_\alpha$ is the multilinear fractional integral operator.

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In his paper [1] K. Moen derived a sufficient condition guaranteeing the two-weight inequality

$$\left(\int_{\mathbb{R}^n} |\mathcal{N}_\alpha(\vec{f})(x)|^q v(x)dx\right)^{1/q} \leq C \prod_{i=1}^m \left(\int_{\mathbb{R}^n} |f_i(y_i)|^{p_i} w_i(y_i)dy_i\right)^{1/p_i}, \quad p \leq q, \tag{2}$$

where $1/p = \sum_{k=1}^m 1/p_k$, for the multilinear Riesz potential $\mathcal{N}_\alpha = \mathcal{I}_\alpha$. Taking $w_i \equiv$ *const* in (2), then that condition is simultaneously necessary and sufficient if the weight v satisfies some additional conditions, e.g., belongs to the Muckenhoupt's A_∞ class. We prove inequality (1) under a D. Adams [2] type (see also [3]) condition without any additional assumption on v.

It should be emphasized that there are various known methods to derive the inequality

$$\left(\int_{\mathbb{R}^n} |I_\alpha(f)(x)|^q v(x)dx\right)^{1/q} \leq C\left(\int_{\mathbb{R}^n} |f(x)|^p dx\right)^{1/p}, \quad p < q, \tag{3}$$

for the linear fractional integral operator I_α.

Inequality (3) is the special case of the two-weight inequality

$$\left(\int_{\mathbb{R}^n} |I_\alpha(f)(x)|^q v(x)dx\right)^{1/q} \leq C\left(\int_{\mathbb{R}^n} |f(x)|^p w(x)\,dx\right)^{1/p}. \tag{4}$$

In [3] the author used the Marcinkiewicz interpolation theorem to get (3). The two-weight problem for I_α was solved by E. Sawyer (see [4,5]) under conditions involving the operator I_α itself. A complete characterization of the two-weight weak-type inequality

$$v\left(\{x \in \mathbb{R}^n : |I_\alpha(f)(x)| > \lambda\}\right) \leq \frac{c}{\lambda^q}\left(\int_{\mathbb{R}^n} |f(x)|^p w(x)dx\right)^{q/p}, \quad p < q, \tag{5}$$

was given by M. Gabidzashvili and V. Kokilashvili under transparent conditions on weights (see, e.g., [6]). The latter results together with E. Sawyer's condition gives a simple (integral) complete characterization of (4) (see [7]).

A proof of (3) is also known using Welland's [8] trick (see, e.g., [9, Ch. 6]).

Finally, we mention that the trace inequality characterization for I_α in the diagonal case was derived by V. Mazy'a and I. Verbitsky [10] in terms of a pointwise condition on a weight (measure).

Our proof for (1) is different from the methods described above and is based on an L. Hedberg's [11] type inequality. In our case we estimate the potential pointwisely by the fractional maximal function (instead of the Hardy–Littlewood maximal operator) and then the derived inequality is used to establish the desired result.

2. Preliminaries

Multilinear fractional integrals were introduced in the papers by L. Grafakos [12], C. Kenig and E. Stein [13], and L. Grafakos and N. Kalton [14]. In particular, these works deal with the operator

$$B_\alpha(f,g)(x) = \int_{\mathbb{R}^n} \frac{f(x+t)g(x-t)}{|t|^{n-\alpha}}dt, \quad 0 < \alpha < n,\ x \in \mathbb{R}^n.$$

In the mentioned papers it was proved that if $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$, where $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, then B_α is bounded from $L^{p_1} \times L^{p_2}$ to L^q.

As a tool to understand B_α, the operators

$$\mathcal{I}_\alpha(\vec{f})(x) = \int_{(\mathbb{R}^n)^m} \frac{f_1(y_1)\cdots f_m(y_m)}{(|x-y_1|+\cdots+|x-y_m|)^{mn-\alpha}}d\vec{y},$$

where $x \in \mathbb{R}^n, 0 < \alpha < nm, \vec{f} := (f_1,\ldots,f_m), \vec{y} := (y_1,\ldots,y_m)$, were studied as well. The corresponding maximal operator is given by (see [1])

$$\mathcal{M}_\alpha(\vec{f})(x) = \sup_{Q \ni x} \prod_{i=1}^m \frac{1}{|Q|^{1-\alpha/(nm)}} \int_Q |f_i(y_i)|dy_i, \quad 0 \leq \alpha < mn,$$

where $|Q|$ denotes the volume of the cube Q with sides parallel to the coordinate axes.

This operator for $\alpha = 0$ was introduced and studied in [15].

It can be immediately checked that

$$\mathcal{I}_\alpha(f_1,\ldots,f_m) \geq c_n \mathcal{M}_\alpha(f_1,\ldots,f_m), \quad f_i \geq 0,\ i = 1,\ldots,m.$$

In the sequel the following notation will be used:

$$\vec{p} := (p_1,\ldots,p_m), \qquad \vec{w} := (w_1,\ldots,w_m),$$