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A critical Kirchhoff type problem involving a nonlocal operator

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ABSTRACT

In this paper we show the existence of non-negative solutions for a Kirchhoff type problem driven by a nonlocal integrodifferential operator, that is

$$-M(\|u\|_Z^2)\mathcal{L}_K u = \lambda f(x,u) + |u|^{2^*-2}u \quad \text{in } \Omega, \quad u = 0 \quad \text{in } \mathbb{R}^n \setminus \Omega$$

where $\mathcal{L}_K$ is an integrodifferential operator with kernel K, Ω is a bounded subset of $\mathbb{R}^n$, M and f are continuous functions, $\|\cdot\|_Z$ is a functional norm and 2^* is a fractional Sobolev exponent.

1. Introduction

In this paper we deal with the following problem

$$\begin{cases} -M\left(\displaystyle\int_{\mathbb{R}^{2n}} |u(x)-u(y)|^2 K(x-y)\,dx\,dy\right) \mathcal{L}_K u = \lambda f(x,u) + |u|^{2^*-2}u & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega \end{cases} \tag{1}$$

where $n > 2s$ with $s \in (0,1)$, $2^* = 2n/(n-2s)$, λ is a positive parameter, $\Omega \subset \mathbb{R}^n$ is an open bounded set, M and f are two continuous functions whose properties will be introduced later and $\mathcal{L}_K$ is a nonlocal operator defined as follows:

$$\mathcal{L}_K u(x) = \frac{1}{2}\int_{\mathbb{R}^n} (u(x+y) + u(x-y) - 2u(x))K(y)\,dy,$$

for all $x \in \mathbb{R}^n$, where $K : \mathbb{R}^n \setminus \{0\} \to (0,+\infty)$ is a measurable function with the property that

$$\begin{aligned} &\text{there exists } \theta > 0 \text{ and } s \in (0,1) \quad \text{such that} \\ &\theta|x|^{-(n+2s)} \le K(x) \le \theta^{-1}|x|^{-(n+2s)} \quad \text{for any } x \in \mathbb{R}^n \setminus \{0\}. \end{aligned} \tag{2}$$

It is immediate to observe that $mK \in L^1(\mathbb{R}^n)$ by setting $m(x) = \min\left\{|x|^2, 1\right\}$. A typical example for K is given by $K(x) = |x|^{-(n+2s)}$. In this case problem (1) becomes

$$\begin{cases} M\left(\displaystyle\int_{\mathbb{R}^{2n}} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy\right)(-\Delta)^s u = \lambda f(x,u) + |u|^{2^*-2}u & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases} \tag{3}$$

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where $-(-\Delta)^s$ is the fractional Laplace operator which (up to normalization factors) may be defined as

$$-(-\Delta)^s u(x) = \frac{1}{2}\int_{\mathbb{R}^n} \frac{u(x+y)+u(x-y)-2u(x)}{|y|^{n+2s}}\,dy$$

for $x \in \mathbb{R}^n$ (see [1] and references therein for further details on the fractional Laplacian and on the fractional Sobolev space $H^s(\mathbb{R}^n)$).

Problems (1) and (3) have a variational nature and the natural space where finding solutions for them is the homogeneous fractional Sobolev space $H^s_0(\Omega)$ (see [1]). In order to study (1) and (3) it is important to encode the 'boundary condition' $u = 0$ in $\mathbb{R}^n \setminus \Omega$ (which is different from the classical case of the Laplacian) in the weak formulation, by considering also that in the norm $\|u\|_{H^s(\mathbb{R}^n)}$ the interaction between Ω and $\mathbb{R}^n \setminus \Omega$ gives positive contribution. The functional space that takes into account these boundary conditions will be denoted by Z and it was introduced in [2] in the following way.

First, we denote by X the linear space of Lebesgue measurable functions $u : \mathbb{R}^n \to \mathbb{R}$ such that

$$\text{the map } (x, y) \mapsto (u(x) - u(y))^2 K(x-y) \quad \text{is in } L^1(Q, dxdy),$$

where $Q := \mathbb{R}^{2n} \setminus (\mathcal{C}\Omega \times \mathcal{C}\Omega)$. The space X is endowed with the norm

$$\|u\|_X = \left(\|u\|_{L^2(\Omega)} + \int_Q |u(x)-u(y)|^2 K(x-y)\,dx\,dy\right)^{1/2}. \tag{4}$$

It is immediate to observe that bounded and Lipschitz functions belong to X, thus X is not reduced to $\{0\}$ (see [3,4] for further details on space X). Now, the functional space Z denotes the closure of $C_0^\infty(\Omega)$ in X. By [2, Lemma 4], the space Z is an Hilbert space which can be endowed with the norm defined as

$$\|u\|_Z = \left(\int_Q |u(x)-u(y)|^2 K(x-y)\,dx\,dy\right)^{1/2}. \tag{5}$$

Note that in (4) and (5) the integrals can be extended to all $\mathbb{R}^{2n}$, since $u = 0$ a.e. in $\mathbb{R}^n \setminus \Omega$.

In view of our problem, we suppose that $M : \mathbb{R}^+ \to \mathbb{R}^+$ verifies the following conditions:

$$M \text{ is an increasing and continuous function;} \tag{6}$$

$$\text{there exists } m_0 > 0 \quad \text{such that } M(t) \geq m_0 = M(0) \text{ for any } t \in \mathbb{R}^+. \tag{7}$$

A typical example for M is given by $M(t) = m_0 + tb$ with $b \geq 0$.

Also, we assume that $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a continuous function that satisfies:

$$\lim_{|t|\to 0} \frac{f(x,t)}{t} = 0, \quad \text{uniformly in } x \in \Omega; \tag{8}$$

$$\text{there exists } q \in (2, 2^*) \quad \text{such that } \lim_{|t|\to\infty} \frac{f(x,t)}{t^{q-1}} = 0 \text{ uniformly in } x \in \Omega; \tag{9}$$

$$\begin{aligned}&\text{there exists } \sigma \in (2, 2^*) \quad \text{such that for any } x \in \Omega \text{ and } t > 0\\ &0 < \sigma F(x,t) = \sigma \int_0^t f(x,s)\,ds \leq t f(x,t).\end{aligned} \tag{10}$$

Moreover, since we intend to find non-negative solutions, we assume this further condition for f

$$f(x,t) = 0 \quad \text{for any } x \in \Omega \text{ and } t \leq 0. \tag{11}$$

An example of a function satisfying the conditions (8)–(11) is given by

$$f(x,t) = \begin{cases} 0 & \text{if } t \leq 0,\\ a(x)t^{q-1} & \text{if } 0 < t < 1,\\ a(x)t^{q_1-1} & \text{if } t \geq 1,\end{cases}$$

with $2 < q_1 < q$, $a \in L^\infty(\Omega)$ and $a(x) > 0$ for any $x \in \Omega$.

The weak formulation of (1) is given by the following problem

$$\begin{cases} M(\|u\|_Z^2)\displaystyle\int_{\mathbb{R}^{2n}} (u(x)-u(y))(\varphi(x)-\varphi(y))K(x-y)\,dx\,dy\\ \quad = \lambda \displaystyle\int_\Omega f(x,u(x))\varphi(x)\,dx + \int_\Omega |u(x)|^{2^*-2}u(x)\varphi(x)\,dx \qquad \forall \varphi \in Z\\ u \in Z.\end{cases} \tag{12}$$