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Remarks on the global regularity of the two-dimensional magnetohydrodynamics system with zero dissipation

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ABSTRACT

We study the two-dimensional magnetohydrodynamics system with generalized dissipation and diffusion in terms of fractional Laplacians. It is known that the classical magnetohydrodynamics system with full Laplacians in both dissipation and diffusion terms admits a unique global strong solution pair. Making use of the special structure of the system in the two-dimensional case, we show in particular that the solution pair remains smooth when we have zero dissipation but only magnetic diffusion with its power of the fractional Laplacian $\beta > \frac{3}{2}$.

1. Introduction and statement of results

We study the following magnetohydrodynamics (MHD) system:

$$\begin{cases} \frac{\partial u}{\partial t} + (u\cdot\nabla)u - (b\cdot\nabla)b + \nabla\pi + \nu\Lambda^{2\alpha}u = 0 \\ \frac{\partial b}{\partial t} + (u\cdot\nabla)b - (b\cdot\nabla)u + \eta\Lambda^{2\beta}b = 0 \\ \nabla\cdot u = \nabla\cdot b = 0, \quad (u,b)(x,0) = (u_0,b_0)(x) \end{cases} \tag{1}$$

where $u : \mathbb{R}^N\times\mathbb{R}^+ \mapsto \mathbb{R}^N$ represents the velocity vector field, $b : \mathbb{R}^N\times\mathbb{R}^+ \mapsto \mathbb{R}^N$ the magnetic vector field, $\pi : \mathbb{R}^N\times\mathbb{R}^+ \mapsto \mathbb{R}$ the pressure scalar field and $\nu, \eta \geq 0$ are the kinematic viscosity and diffusivity constants respectively. We also let $\hat{f}(\xi)$ denote the Fourier transform of f; i.e.

$$\hat{f}(\xi) = \int_{\mathbb{R}^N} f(x)e^{-ix\cdot\xi}dx$$

and defined a fractional Laplacian operator $\Lambda^{2\gamma}$ with $\gamma \in \mathbb{R}$ to have the Fourier symbol of $|\xi|^{2\gamma}$; that is,

$$\widehat{\Lambda^{2\gamma}f}(\xi) = |\xi|^{2\gamma}\hat{f}(\xi).$$

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In case $N = 2, 3, \nu, \eta > 0, \alpha = \beta = 1$, the MHD system possesses at least one global L^2 weak solution for any initial data pair $(u_0, b_0) \in L^2(\mathbb{R}^N) \times L^2(\mathbb{R}^N)$; in case $N = 2$, in fact the solution is unique (cf. [1]).

In order to discuss the previous results on strong solutions and better understand the importance of the lower bounds for the two parameters $\alpha, \beta > 0$ when $\nu, \eta > 0$, let us recall the notion of criticality in a simple setting. First, it can be shown that the solution pair to (1) with $\alpha = \beta = \gamma$ has the rescaling properties that if $(u(x, t), b(x, t))$ solves the system, then so does $(u_\lambda(x, t), b_\lambda(x, t))$ with $\lambda \in \mathbb{R}^+$ where

$$u_\lambda(x, t) = \lambda^{2\gamma-1}u(\lambda x, \lambda^{2\gamma} t), \qquad b_\lambda(x, t) = \lambda^{2\gamma-1}b(\lambda x, \lambda^{2\gamma} t), \quad \gamma \in \mathbb{R}^+.$$

As we show in (4), the solution pair (u, b) to (1) has the global bounds on the L^2-norm and it can be shown that $\gamma = \frac{1}{2} + \frac{N}{4}$ implies

$$\|u_\lambda(\cdot, t)\|_{L^2(\mathbb{R}^N)} = \|u(\cdot, \lambda^{2\gamma} t)\|_{L^2(\mathbb{R}^N)}, \qquad \|b_\lambda(\cdot, t)\|_{L^2(\mathbb{R}^N)} = \|b(\cdot, \lambda^{2\gamma} t)\|_{L^2(\mathbb{R}^N)}.$$

With this in mind, we call the case $\nu, \eta > 0, \alpha \geq \frac{1}{2} + \frac{N}{4}, \beta \geq \frac{1}{2} + \frac{N}{4}$ the critical case and in such a case, the existence of the unique global strong solution pair has been shown (cf. [2]).

Some numerical analysis results (e.g. [3,4]) indicate a more dominant role played by the velocity vector field in preserving the regularity of the solution pair. Moreover, starting from the works of [5,6], we have also seen various regularity criteria of the MHD system in terms of only the velocity vector field (e.g. [7–14]). This is largely due to the fact that upon taking H^1-estimates of u and b, every nonlinear term involves u while not necessarily b. With this in mind, following the work of [15], the author in [16] showed that even in logarithmically super-critical case the system (1) still admits a unique global strong solution pair. That is, the author replaced the dissipative term of $\nu\Lambda^{2\alpha}u$ and the diffusive term of $\eta\Lambda^{2\beta}b$ by $\nu\mathcal{L}_1^2 u$ and $\eta\mathcal{L}_2^2 b$ respectively where $\mathcal{L}_i, i = 1, 2$ are defined to have the Fourier symbols of $m_i(\xi), i = 1, 2$ satisfying the following lower bounds:

$$\widehat{\mathcal{L}_1 u}(\xi) = m_1(\xi)\hat{u}(\xi), \qquad \widehat{\mathcal{L}_2 b}(\xi) = m_2(\xi)\hat{b}(\xi)$$

and

$$m_1(\xi) \geq \frac{|\xi|^\alpha}{g_1(\xi)}, \qquad m_2(\xi) \geq \frac{|\xi|^\beta}{g_2(\xi)}, \qquad \alpha \geq \frac{1}{2} + \frac{N}{4}, \quad \beta > 0,\ \alpha + \beta \geq 1 + \frac{N}{2}$$

with $g_i \geq 1, i = 1, 2$ being radially symmetric, non-decreasing functions.

The endpoint case $\nu > 0, \eta = 0, \alpha = 1 + \frac{N}{2}$ was also completed recently in [17] (cf. also [18] for further generalization).

On the other hand, in case $N = 2$, it is well-known that the Euler equation, the Navier–Stokes system with no dissipation, admits a unique global strong solution. This is due to the fact that upon taking a curl, the vorticity becomes a conserved quantity. In the case of the MHD system, upon taking a curl and L^2-estimate of the resulting system, every nonlinear term has b involved. Exploiting this observation and divergence-free conditions, the authors in [19] showed that in case $N = 2$, full Laplacians in both dissipation and magnetic diffusion are not necessary for the solution to remain smooth; rather, only a mix of partial dissipation and diffusion in the order of two derivatives suffices. In this paper we make further observation in case $N = 2$:

Theorem 1.1. *Let $N = 2, \nu = 0, \eta > 0, \alpha = 0, \beta > \frac{3}{2}$. Then for all initial data pair $(u_0, b_0) \in H^s(\mathbb{R}^2) \times H^s(\mathbb{R}^2), s \geq 1 + 2\beta$, there exists a unique global strong solution pair (u, b) to (1) such that*

$$u \in C([0, \infty); H^s(\mathbb{R}^2))$$
$$b \in C([0, \infty); H^s(\mathbb{R}^2)) \cap L^2([0, \infty); H^{s+\beta}(\mathbb{R}^2)).$$

Theorem 1.2. *Let $N = 2, \nu, \eta > 0, \alpha \in \left(0, \frac{1}{2}\right), \beta \in \left(\frac{5}{4}, \frac{3}{2}\right]$ such that $\alpha + 2\beta > 3$. Then for all initial data pair $(u_0, b_0) \in H^s(\mathbb{R}^2) \times H^s(\mathbb{R}^2), s \geq 1 + 2\beta$, there exists a unique global strong solution pair (u, b) to (1) such that*

$$u \in C([0, \infty); H^s(\mathbb{R}^2)) \cap L^2([0, \infty); H^{s+\alpha}(\mathbb{R}^2))$$
$$b \in C([0, \infty); H^s(\mathbb{R}^2)) \cap L^2([0, \infty); H^{s+\beta}(\mathbb{R}^2)).$$

Remark 1.1. (1) Our proof was inspired partially from the work of [19,20,8]. We note that making use of the structure of the partial differential equation has proven to be useful in other cases as well (e.g. [21]).

(2) While this paper was being prepared, the work by [22] appeared. In their work, it is shown that in particular if $\alpha = 0$, then $\beta > 2$ is required (see Theorem 1 and Remark 1 of [22]) while our Theorem 1.1 shows that $\beta > \frac{3}{2}$ suffices. We also independently obtained Theorem A.1; this is no longer a new result and thus we placed this in the Appendix because its proof is immediate and very simple. The hypothesis of Theorem A.1 allows $\alpha \geq \frac{1}{2}$ rather than $\alpha = 0$ as in Theorem 1.1. As will be discussed, a complete lack of dissipation makes the analysis significantly more difficult in the latter case.

(3) There are ways to obtain different initial regularity in various space of functions; we chose to state the above for simplicity. We also refer readers to [19] where the authors considered the case $N = 2, \nu = 0, \eta > 0, \beta = 1$ and showed the existence of weak solution pair and regularity criteria for its global regularity and uniqueness (cf. also [2]).

(4) To extend such a type of result to higher dimension, it seems to require a new idea. As indicated in the work of [16,17], in higher dimension, dissipation seems to be crucial in preserving the regularity of the solution pair.

In the Preliminary section, let us briefly set up notations and state key lemmas; thereafter, we prove our theorems.