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A well posedness result for nonlinear viscoelastic equations with memory

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A B S T R A C T

We establish an existence, uniqueness and continuous dependence result for the weak solutions to the nonlinear viscoelastic equation with hereditary memory on a bounded three-dimensional domain

$$|\partial_t u|^\rho \partial_{tt} u - \Delta \partial_{tt} u + \gamma(-\Delta)^\theta \partial_t u - \alpha \Delta u + \int_0^\infty \mu(s) \Delta u(t-s)\, ds + f(u) = h$$

with Dirichlet boundary conditions. In particular, the parameter ρ belongs to the interval $[0, 4]$, the value 4 being critical for the Sobolev embeddings, while f can reach the critical polynomial order 5.

1. Introduction

Given a bounded domain $\Omega \subset \mathbb{R}^3$ with smooth boundary $\partial\Omega$, we denote by

$$A = -\Delta$$

the Dirichlet operator with domain

$$\mathrm{dom}(A) = H^2(\Omega) \cap H_0^1(\Omega) \Subset L^2(\Omega).$$

Let $\rho \in [0, 4]$, $\theta \in [0, 1]$, $\gamma \geq 0$ and $\alpha > 0$ be fixed parameters. For $t \in \mathbb{R}^+ = (0, \infty)$, we consider the equation

$$|\partial_t u|^\rho \partial_{tt} u + A \partial_{tt} u + \gamma A^\theta \partial_t u + \alpha A u - \int_0^\infty \mu(s) A u(t-s)\, ds + f(u) = h \tag{1.1}$$

in the unknown variable $u = u(\boldsymbol{x}, t) : \Omega \times \mathbb{R} \to \mathbb{R}$ subject to the Dirichlet boundary condition

$$u(\boldsymbol{x}, t)_{|\boldsymbol{x} \in \partial\Omega} = 0.$$

The model is supplemented with the initial conditions (the dependence on $\boldsymbol{x}$ is omitted)

$$u(0) = u_0, \qquad \partial_t u(0) = v_0, \qquad u(-s)_{|s \in \mathbb{R}^+} = \psi_0(s), \tag{1.2}$$

where $u_0, v_0 : \Omega \to \mathbb{R}$ and $\psi_0 : \Omega \times \mathbb{R}^+ \to \mathbb{R}$ are given functions, the latter accounting for the initial past history of u.

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Here, the time-independent external force h belongs to the dual space $H^{-1}(\Omega)$ of $H_0^1(\Omega)$, while the locally Lipschitz nonlinearity f, with $f(0) = 0$, fulfills the critical growth restriction

$$|f(u) - f(v)| \le c|u - v|(1 + |u|^4 + |v|^4), \tag{1.3}$$

along with the dissipation condition

$$\liminf_{|u|\to\infty} \frac{f(u)}{u} > -\lambda_1, \tag{1.4}$$

where $\lambda_1 > 0$ is the first eigenvalue of A. Finally, the convolution (or memory) kernel μ is a nonnegative, nonincreasing, piecewise absolutely continuous function on $\mathbb{R}^+$ of total mass

$$\int_0^\infty \mu(s)\,\mathrm{d}s = \kappa \in [0, \alpha).$$

Without loss of generality, we may take

$$\alpha - \kappa = 1. \tag{1.5}$$

In particular, μ is allowed to exhibit (even infinitely many) jumps, and can be unbounded about the origin.

Remark 1.1. The degenerate case $\mu \equiv 0$, corresponding to the partial differential equation

$$|\partial_t u|^\rho \partial_{tt} u + A\partial_{tt} u + \gamma A^\theta \partial_t u + Au + f(u) = h,$$

is included in our analysis.

Problem (1.1), featuring the nonlinear term

$$|\partial_t u|^\rho \partial_{tt} u,$$

provides a generalization, accounting for memory effects in the material, of equations of the form

$$\varrho(\partial_t u)\partial_{tt} u + A\partial_{tt} u + Au = 0.$$

Such PDEs arise in mechanics, in the description of the vibrations of thin rods whose material density $\varrho(\partial_t u)$ is not constant (see e.g. [1]). The model under consideration has been the object of intensive investigations in the last decade, mainly in its simplified Volterra version

$$|\partial_t u|^\rho \partial_{tt} u + A\partial_{tt} u + \gamma A^\theta \partial_t u + \alpha Au - \int_0^t \mu(s)Au(t-s)\,\mathrm{d}s = 0, \tag{1.6}$$

which turns out to be a particular instance of (1.1), as shown in the next Section 3.

The first result concerning (1.6) appears in [2], where the global existence of weak solutions is established for $\theta = 1$ and $\gamma \ge 0$, provided that

$$\rho \le 2.$$

Besides, assuming $\gamma > 0$ and an exponentially decaying memory kernel μ, the authors demonstrate the exponential decay of solutions. However, since no uniqueness is proved, such a result holds only for those trajectories that can be obtained as limits in the Galerkin approximation scheme.

After [2], the study of the longterm properties of the (Galerkin) solutions to (1.6) has been tackled in several works (e.g. [3–12]), with various decay hypotheses on μ. Still, in the above-mentioned papers, the restriction $\rho \le 2$ is always assumed. Indeed, all of them refer to [2] for the existence result, with the only exception of [4], which actually recasts the argument of [2].

On the contrary, the uniqueness issue has never been addressed until the very recent article [13], dealing with the more general model (1.1) with $\theta = 1$, in presence of a nonlinearity $f(u)$ of cubic growth satisfying $uf(u) \ge 0$. There, leaning on [2], the authors show the existence of solutions for $\rho \le 2$, and they prove a continuous dependence (whence uniqueness) result under the additional request that the map

$$v \mapsto |v|^\rho$$

be differentiable at zero, which introduces the further restriction

$$\rho > 1.$$

The aim of our paper is to establish the ultimate well-posedness result for (1.1)–(1.2), ensuring existence (see Section 4) and continuous dependence from the initial data (see Section 5) for the most general admissible nonlinearity $f(u)$, as well as for ρ belonging to the whole meaningful range [0, 4].

Once well-posedness is attained, the study of the asymptotic properties of the solution semigroup becomes a meaningful and interesting question, that will be possibly addressed in forthcoming papers.