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# Hyperbolic mean curvature flow in Minkowski space

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ABSTRACT

We study the hyperbolic mean curvature flow (HMCF) of graphs in Minkowski space. A quasilinear wave equation is derived and studied for the motion of smooth immersed spacelike acausal closed curves under HMCF. Based on this, we investigate the formation of singularities in the motion of these curves. Some blow-up results have been obtained and the estimates on the life-span of the solutions are given. Furthermore, our results show that the curvature of the limit curve become unbounded as $t \to T_{\max}$.



## 1. Introduction

Minkowski space $R^{1,1}$ is the linear space $R^{1+1}$ endowed with the Lorentz metric

$$ds^2 = dx^2 - dy^2.$$

Spacelike curves in $R^{1,1}$ are Riemannian 1-manifolds, having an everywhere lightlike normal vector $\vec{\nu}$ which assume to be future directed and thus satisfy the condition $\langle \vec{\nu}, \vec{\nu} \rangle = -1$. Locally, such curves can be expressed as graphs of functions $y = f(x) : R \mapsto R$ satisfying the spacelike conditions $|f_x| < 1$ for all $x \in R$.

If a family of spacelike embeddings $\gamma_t = \gamma(\cdot, t) : S^1 \mapsto R^{1,1}$ with corresponding curves $M_t = \gamma(S^1, t)$ satisfy the following evolution equation

$$\begin{cases} \dfrac{\partial^2 \gamma}{\partial t^2}(z,t) = k(z,t)\vec{\nu}(z,t) + \rho(z,t)\vec{T}(z,t), \quad \forall\, \gamma : S^1 \times [0,T) \to R^{1,1}, \\ \gamma(z,0) = \gamma_0(z), \quad \dfrac{\partial \gamma}{\partial t}(z,0) = h(z)\vec{N}_0, \end{cases} \tag{1.1}$$

where $k$ denotes the mean curvature of the curve $M_t$, $\vec{\nu}$ is the unit inner normal vector of $M_t$, the function $\rho$ is defined by

$$\rho = -\left\langle \frac{\partial^2 \gamma}{\partial s \partial t}, \frac{\partial \gamma}{\partial t} \right\rangle \tag{1.2}$$

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in which $s$ is the arclength parameter, $\vec{T}$ stands for the unit tangent vector of $\gamma_t$, $\gamma_0$ denotes the initial closed curve, while $h$ and $\vec{v}_0$ are the initial velocity and unit inner normal vector of initial curve $\gamma_0$, respectively. Clearly, the initial velocity is normal to the initial curve, and at the beginning of Section 2 we will show that the flow described by (1.1) is always normal one. On the other hand, it is easy to see that (1.1) is an initial value problem for a system of second-order hyperbolic differential equations. Similar to [1], we can prove the following theorem.

**Theorem A** (*Local Existences and Uniqueness*)**.** *Let $\gamma_0$ be a smooth spacelike acausal closed curve immersion of $S^1$ into $R^{1,1}$, and $\frac{\partial \gamma}{\partial t}(z, 0)$ be an initial velocity. Then there exist a positive $T$ and a family of smooth spacelike acausal closed curves $\gamma(\cdot, t)$ with $t \in [0, T)$ such that the Cauchy problem (1.1) admits a unique smooth solution $\gamma(\cdot, t)$ on $S^1$, provided that $h(z)$ is a smooth function on $S^1$.*

Traditionally, mean curvature flow (MCF) has been extensively studied in Euclidean space; see [2–7], and the references therein, while in Minkowski space, MCF was studied in [8,9] for compact hypersurfaces and in [10,11] for noncompact hypersurfaces. The method of MCF was used [8,9] to construct spacelike hypersurfaces with prescribed mean curvature, which, as it is well-known, have played important roles in studying Lorentzian manifolds. In 2001, Huisken and Ilmanen introduced the inverse mean curvature flow (IMCF), developed a theory of weak solutions of the IMCF and used this theory to prove successfully the Riemannian Penrose inequality which plays an important role in general relativity (see [12]).

However, to our knowledge, there is very few hyperbolic versions of mean curvature flow. The hyperbolic version of mean curvature flow is important in both mathematics and applications, and has attracted many mathematicians to study it (e.g., [13–16]). Recently, Kong and Liu introduced the hyperbolic geometric flow which is an attempt to solve some problems arising from differential geometry and theoretical physics (in particular, general relativity). The hyperbolic geometric flow is a very natural tool to understand the wave character of the metrics, wave phenomenon of the curvatures, the evolution of manifolds and their structures (see [17,18,1,19–23]). Contrast to the hyperbolic mean curvature flows studied in [1,21,24], hyperbolic gauss curvature flow [25] is proposed for convex hypersurfaces. The equation satisfied by the graph of the hypersurface under this flow gives rise to a new class of fully nonlinear Euclidean invariant hyperbolic equations.

In this paper we particularly investigate the formation of singularities of the evolution of convex closed spacelike curves under hyperbolic mean curvature flow in the Minkowski space $R^{1,1}$. We shall prove that the smooth solution of the Cauchy problem (1.1) will, in general, blow up in finite time, provided that the perimeter of the initial closed curve and the initial velocity is suitably small, or the initial data satisfies some additional (but not smallness) assumptions. Furthermore, our results show that the curvature of the limit curve become unbounded as $t \to T_{\max}$. See Section 3 for the detailed blowup results.

The paper is organized as follows. In Section 2, we derive a second-order quasilinear wave equation, and by constructing the Riemann invariants we reduce the wave equation to a reducible quasilinear hyperbolic system of first order, based on this, we analyze some interesting properties enjoyed by this system. The main results are stated in Section 3. Section 4–5 are devoted to the proof of the main results.

## 2. Basic equations: derivation and properties

We first illustrate the flow described by (1.1) is normal one.

In fact, noting

$$\begin{aligned}\frac{\partial}{\partial t}\left\langle \frac{\partial \gamma}{\partial t}, \frac{\partial \gamma}{\partial z}\right\rangle &= \left\langle \frac{\partial^2 \gamma}{\partial t^2}, \frac{\partial \gamma}{\partial z}\right\rangle + \left\langle \frac{\partial \gamma}{\partial t}, \frac{\partial^2 \gamma}{\partial z \partial t}\right\rangle = \left\langle \rho \vec{T}, \frac{\partial \gamma}{\partial z}\right\rangle + \left\langle \frac{\partial \gamma}{\partial t}, \frac{\partial^2 \gamma}{\partial z \partial t}\right\rangle \\ &= -\left\langle \frac{\partial F}{\partial t}, \frac{\partial^2 \gamma}{\partial z \partial t}\right\rangle + \left\langle \frac{\partial F}{\partial t}, \frac{\partial^2 \gamma}{\partial z \partial t}\right\rangle = 0,\end{aligned}$$

we have

$$\left\langle \frac{\partial \gamma}{\partial t}, \frac{\partial \gamma}{\partial z}\right\rangle (z, t) = \left\langle \frac{\partial \gamma}{\partial t}, \frac{\partial \gamma}{\partial z}\right\rangle (z, 0) = 0.$$

This implies that, if the initial velocity field is normal to the initial curve, then this property is preserved during the evolution. Therefore, noting the third equation in (1.1) we observe that the flow under consideration is normal one.

Locally, such curves can be expressed as graphs of functions $u(x, t) : R \mapsto R$ satisfying the spacelike conditions $|u_x| < 1$ for all $x \in R$. Then we can write $\gamma$ as

$$\gamma(z, t) = (x, u(x, t)), \quad \forall x \in R.$$

Thus, we have

$$\frac{\partial \gamma}{\partial t} = \frac{dx}{dt}(1, u_x) + \left(0, \frac{\partial u}{\partial t}\right). \tag{2.1}$$