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Global well-posedness of strong solutions to the magnetohydrodynamic equations of compressible flows

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ABSTRACT

This paper is concerned with the global well-posedness of strong solutions to the compressible MHD equations in a bounded domain $\Omega \subset \mathbb{R}^3$. We first establish the local existence and uniqueness of strong solutions with arbitrary initial data and external force field. Based on the local existence result and some a priori estimates, the global existence and large time behavior of strong solutions are proved, under the assumptions that the initial data and external force field are small enough. Furthermore, the existence of the time periodic solution is also obtained when the external force field is time periodic.

1. Introduction

In the present paper, we consider the MHD equations of three-dimensional compressible flows in the isentropic case [1–4]:

$$\begin{cases} \rho_t + \operatorname{div}(\rho\mathbf{u}) = 0 & \text{in } Q_T, \\ \rho(\mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u}) - \mu\Delta\mathbf{u} - \zeta\nabla\operatorname{div}\mathbf{u} + \nabla p = (\nabla\times\mathbf{H})\times\mathbf{H} + \rho\mathbf{b} & \text{in } Q_T, \\ \mathbf{H}_t - \nabla\times(\mathbf{u}\times\mathbf{H}) = -\nabla\times(\nu\nabla\times\mathbf{H}) \quad \text{and} \quad \operatorname{div}\mathbf{H} = 0 & \text{in } Q_T, \\ \mathbf{u}|_{\partial\Omega} = \mathbf{H}|_{\partial\Omega} = \mathbf{0}, & \text{on } \Sigma_T, \\ (\rho, \mathbf{u}, \mathbf{H})|_{t=0} = (\rho_0, \mathbf{u}_0, \mathbf{H}_0) \quad \text{with } \operatorname{div}\mathbf{H}_0 = 0 & \text{in } \Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^3$ is a bounded domain, $Q_T \equiv (0, T) \times \Omega$, $\Sigma_T \equiv (0, T) \times \partial\Omega$, $0 < T \leq \infty$, $\rho = \rho(t, x)$ denotes the density of the fluid, $\mathbf{u} = \mathbf{u}(t, x) \in \mathbb{R}^3$ the velocity, $\mathbf{H} = \mathbf{H}(t, x) \in \mathbb{R}^3$ the magnetic field, $\mathbf{b} = \mathbf{b}(t, x) \in \mathbb{R}^3$ the (assigned) external force field and $p = p(\rho)$ is the pressure, which is assumed to be a known increasing function of ρ. The viscosity coefficients μ, ζ satisfy $\mu > 0$, $\zeta = \mu + \lambda$ with $\frac{2}{3}\mu + \lambda > 0$ and $\nu > 0$ is the magnetic diffusivity acting as a magnetic diffusion coefficient of the magnetic field. Finally, $\rho_0 = \rho_0(x) > 0$, $\mathbf{u}_0 = \mathbf{u}_0(x)$ and $\mathbf{H}_0 = \mathbf{H}_0(x)$ with $\operatorname{div}\mathbf{H}_0 = 0$ are the initial density, initial velocity and initial magnetic field, respectively.

Usually, we refer to $(1.1)_1$ as the continuity equation and $(1.1)_2$ as the momentum balance equations. It is well-known that the electromagnetic fields are governed by the Maxwell equations. In MHD, the displacement current can be neglected ([3,4]). As a consequence, the equations $(1.1)_3$ are called the induction equations, and the electric field $\mathbf{E}$ can be written in terms of the magnetic field $\mathbf{H}$ and the velocity $\mathbf{u}$,

$$\mathbf{E} = \nu\nabla\times\mathbf{H} - \mathbf{u}\times\mathbf{H}.$$

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Although the electric field **E** does not appear in (1.1), it is indeed induced according to the above relation by the moving conductive flow in the magnetic field.

In this paper, we are mainly concerned with the global well-posedness of the strong solution to the three-dimensional MHD equations (1.1). The existence of time periodic solutions is also considered.

In the recent years, there have been a lot of studies on MHD by physicists and mathematicians because of its physical importance, complexity, rich phenomena, and mathematical challenges; see [5–8] and the references cited therein. In particular, the one-dimensional problem has been studied in many papers, see, for examples, [5,9–11,6,8] and the references cited therein. However, a number of fundamental problems for MHD are still open, for example, even in the one-dimensional case, the global existence of classical solutions to the full perfect MHD equations with large data remains unsolved when all the viscosity, heat conductivity, and diffusivity coefficients are constants. The reason is that the presence of the magnetic field and its interaction with the hydrodynamic motion in MHD flows of large oscillation cause serious difficulties.

At the absence of external force, i.e., $\mathbf{b} \equiv \mathbf{0}$, on one hand, there have been some results concerning the Cauchy problem for $(1.1)_1$–$(1.1)_3$. Li and Yu [7] established the global classical solution and presented the optimal decay rate of the solution when the initial data are small perturbations of some given constant state. The three-dimensional compressible MHD isentropic flow with zero magnetic diffusivity was studied by Li and Wang [12]. They obtained that the existence and uniqueness of local in time strong solution with large initial data. On the other hand, as for the initial–boundary problem (1.1), Hu and Wang in [2] obtained the global existence and large-time behavior of weak solutions to the multi-dimensional isentropic problem (1.1) with $p = a\rho^\gamma (\gamma > \frac{3}{2})$, where all the viscosity coefficients μ, λ, ν are constants. Under the assumption that the initial density may vanish in an open set, Fan and Yu [13] proved the existence and uniqueness of local strong solution to the full MHD equations with different boundary condition for magnetic field **H**. More recently, Tan and Wang [14] utilized the argument in [15] to establish the existence, uniqueness and time-asymptotic stability of time periodic solutions, under the assumptions that the external force is time periodic in $\mathbb{R}^n$ and the space dimension $n \geq 5$. Compared with this severe restriction on space dimension, we shall establish the existence of time periodic solution in a bounded domain $\Omega \subset \mathbb{R}^3$.

The aim of this paper is to establish the global well-posedness of the strong solutions to the compressible MHD flows (1.1) and to prove the existence of time periodic solution in a bounded domain $\Omega \subset \mathbb{R}^3$. To be specific, we first use some estimates on the linearized problems and Schauder's fixed point theorem to yield the existence of local strong solution to (1.1) with large initial data and large external force field **b**. Based on the local existence, some a priori estimates obtained by Valli [16] and some parabolic estimates for magnetic field **H**, the global existence and the stability are proved under the assumptions that the initial data are small perturbations of some given constant state and the external force field **b** is small enough. We remark that the stability result can be viewed as the generalization of the result obtained by Li and Yu [7] to the bounded domain in $\mathbb{R}^3$. Moreover, we prove the existence of time periodic solution provided that the external force field is time periodic and sufficiently small.

Throughout this paper we assume that

$$m \equiv \min_{\bar{\Omega}} \rho_0(x) > 0, \tag{1.2}$$

and we set

$$M \equiv \max_{\bar{\Omega}} \rho_0(x), \qquad \bar{\rho} \equiv \frac{1}{\mathrm{Vol}\Omega} \int_\Omega \rho_0(x)\,\mathrm{d}x > 0. \tag{1.3}$$

We will denote the norm in $H^k(\Omega)$ (the usual Sobolev space) by $\|\cdot\|_k$ for $k+1 \in \mathbb{N}$; the norm in $L^q(0,T;H^k(\Omega))$ by

$$[\cdot]_{q;k;T},$$

for $1 \leq q \leq \infty$, $k+1 \in \mathbb{N}$, $0 < T \leq \infty$; the norm in $L^q(0,T;L^s(\Omega))$ by

$$|\!|\!| \cdot |\!|\!|_{q;s;T},$$

for $1 \leq q \leq \infty$, $1 \leq s \leq \infty$, $0 < T \leq \infty$. The norm in $L^\infty(0,T;X)$ and in $C^0([0,T];X)$ are denoted in the same way. Moreover, $C^0([0,T];X)$ stands for the space of continuous and bounded functions from $\mathbb{R}^+$ to X.

Since the perturbation is around the constant state $(\bar{\rho}, \mathbf{0}, \mathbf{0})$, it is convenient to introduce the new variable

$$\sigma \equiv \rho - \bar{\rho} \tag{1.4}$$

and rewrite problem (1.1) in a new form:

$$\begin{cases} \sigma_t + \mathbf{u}\cdot\nabla\sigma + \sigma\mathrm{div}\mathbf{u} + \bar{\rho}\mathrm{div}\mathbf{u} = 0 & \text{in } Q_T, \\ (\sigma+\bar{\rho})(\mathbf{u}_t + (\mathbf{u}\cdot\nabla)\mathbf{u}) - \mu\Delta\mathbf{u} - \zeta\nabla\mathrm{div}\mathbf{u} + \nabla p(\sigma+\bar{\rho}) = (\nabla\times\mathbf{H})\times\mathbf{H} + (\sigma+\bar{\rho})\,\mathbf{b} & \text{in } Q_T, \\ \mathbf{H}_t - \nabla\times(\mathbf{u}\times\mathbf{H}) = -\nabla\times(\nu\nabla\times\mathbf{H}) \quad \text{and} \quad \mathrm{div}\mathbf{H} = 0 & \text{in } Q_T, \\ \mathbf{u}|_{\partial\Omega} = \mathbf{H}|_{\partial\Omega} = \mathbf{0}, & \text{on } \Sigma_T, \\ (\sigma,\mathbf{u},\mathbf{H})|_{t=0} = (\rho_0 - \bar{\rho}, \mathbf{u}_0, \mathbf{H}_0) \quad \text{with } \mathrm{div}\mathbf{H}_0 = 0 & \text{in } \Omega. \end{cases} \tag{1.5}$$

When there is no electromagnetic field, the system (1.1) reduces to the compressible Navier–Stokes equations. To our knowledge, the first global existence results have been proved by Matsumura and Nishida (see [17,18]). Moreover, by means of some new a-priori estimates, Valli [16] proved the global existence result and a stability result, so as to obtain the existence