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On the nonlinear stability of parallel shear flow in the presence of a coplanar magnetic field

Lanxi Xu *, Wanli Lan

Department of Mathematics, Beijing University of Chemical Technology, Beijing 100029, China

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ABSTRACT

The nonlinear stability of laminar flow, in the x-direction, between two parallel planes in the presence of a coplanar magnetic field has been studied using Lyapunov direct method with either rigid or stress-free boundary planes. By defining a Lyapunov function it is proved that the laminar solutions of the system are nonlinearly unconditionally and asymptotically stable for all Reynolds numbers and magnetic Reynolds numbers if the perturbations are two-dimensional and depend only on y, z and t.

1. Introduction

The most important part of the hydrodynamic stability theory is the investigation of how laminar flow becomes unstable. Of the available methods of stability analysis, the Lyapunov direct method is the best choice for the investigation of nonlinear stability of laminar flow. The advantage of this method is that it gives sufficient conditions for the stability of the laminar flow.

The Lyapunov direct method, also known as the generalized energy method, was first considered in the context of hydrodynamic stability by Serrin [1], and subsequently used by Joseph [2] and other authors (see [3–8], and references therein).

Here we study the nonlinear stability of the motion of an incompressible, homogeneous, viscous and electrically conducting fluid in a horizontal layer, permeated by an imposed uniform magnetic field $\boldsymbol{H}$ coplanar to the layer. This problem has previously been studied by other authors. In particular, in [9] the linear stability has been considered in the case of rigid boundaries for large magnetic viscosity, and in [10,11] the nonlinear energy stability has been studied for one-dimensional and two-dimensional perturbations in the isothermal and non-isothermal cases. In [7] the authors consider the linear and nonlinear stability of the problem considering the stress-free boundary case for three-dimensional perturbations. By choosing a Lyapunov function they showed that the laminar flow is linearly asymptotically exponentially stable if $R_m < \pi^2/8M$. By introducing another Lyapunov function they showed that the basic motion is nonlinearly conditionally stable for all Reynolds numbers if the magnetic Reynolds number R_m satisfies $R_m < \pi^2/8M$, where M is the maximum of the absolute value of the velocity field of the laminar flow.

By applying the Lyapunov direct method, here we examine the nonlinear stability of laminar basic flow of the aforesaid problem for x-independent perturbations. Cases with rigid and stress-free boundary planes are both examined. By taking advantage of the poloidal–toroidal decomposition we can prove the unconditional stability of the laminar basic flow for

* Corresponding author. Tel.: +86 13717822564.
E-mail addresses: xulx@mail.buct.edu.cn, xulx63@l63.com (L. Xu).

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all Reynolds numbers and magnetic Reynolds numbers. Thanks to the decomposition we can define a generalized energy functional with lower-order derivatives than that of Mulone and Salemi, which enables us to consider the problem with either the rigid or stress-free boundary condition.

The paper is organized as follows: Section 2 contains the laminar basic solutions, the perturbation equations, boundary conditions and the poloidal–toroidal decomposition of a solenoidal vector field. In Section 3 the Lyapunov function is given and a nonlinear stability theorem is formulated.

2. Mathematical formulation

Consider an infinite horizontal homogeneous viscous and electrically conducting fluid layer $\mathbb{R}^2 \times (-d, d)$ in a Cartesian reference frame $oxyz$ with unit vectors $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ respectively. The layer is assumed to be parallel to the xoy plane and the velocities of the planes satisfy the boundary conditions: $\boldsymbol{U}(x, y, z, -d) = -V\boldsymbol{i}$ and $\boldsymbol{U}(x, y, z, d) = V\boldsymbol{i}$. We assume that the medium adjoining the fluid is electrically non-conducting and is permeated with a constant magnetic field $\boldsymbol{H} = H_0\boldsymbol{i} + H_1\boldsymbol{j}$, where $H_0, H_1 \in \mathbb{R}, H_0 \neq 0$.

The equations of motion, given in [12], admit the laminar solutions [7]:

$$\begin{aligned}&\boldsymbol{U} = f(z)\boldsymbol{i} \quad f(z) = \frac{k}{2\nu}(d^2 - z^2) + \frac{Vz}{d}\\ &\boldsymbol{H} = H_0\boldsymbol{i} + H_1\boldsymbol{j}\\ &p_1 = -k\rho_0 x + p_0,\end{aligned}$$

where $k, p_0 \in \mathbb{R}$, ν is the kinematic viscosity, and ρ_0 is the constant density.

The non-dimensional equations which govern a perturbation $(\boldsymbol{u}, \boldsymbol{h}, p)$ to the laminar basic solutions $(\boldsymbol{U}, \boldsymbol{H}, p_1)$ are [7]:

$$\begin{cases}\partial_t \boldsymbol{u} = \dfrac{1}{R_e}\Delta\boldsymbol{u} - wf'(z)\boldsymbol{i} - f(z)\partial_x\boldsymbol{u} - \nabla\lambda - \boldsymbol{u}\cdot\nabla\boldsymbol{u} + A_m[H_0\partial_x\boldsymbol{h} + H_1\partial_y\boldsymbol{h} + \boldsymbol{h}\cdot\nabla\boldsymbol{h}]\\ \partial_t \boldsymbol{h} = \dfrac{1}{R_m}\Delta\boldsymbol{h} + h_3 f'(z)\boldsymbol{i} - f(z)\partial_x\boldsymbol{h} - \boldsymbol{u}\cdot\nabla\boldsymbol{h} + H_0\partial_x\boldsymbol{u} + H_1\partial_y\boldsymbol{u} + \boldsymbol{h}\cdot\nabla\boldsymbol{u}\\ \nabla\cdot\boldsymbol{u} = 0\\ \nabla\cdot\boldsymbol{h} = 0\end{cases} \tag{1}$$

in $\mathbb{R}^2 \times (-1, 1) \times (0, +\infty)$, where $\boldsymbol{u} = (u, v, w)$, $\boldsymbol{h} = (h_1, h_2, h_3)$, $\lambda = \frac{p}{\rho_0} + \frac{A_m|\boldsymbol{H}+\boldsymbol{h}|^2}{2}$, $R_e = \frac{v_0 d}{\nu}$ is the Reynolds number, $R_m = \frac{v_0 d}{\eta}$ is the magnetic Reynolds number, $A_m = \frac{Q^2}{R_e R_m}$ with the Chandrasekhar number Q^2 given by $Q^2 = \frac{\mu H^2 d^2}{\rho_0 \nu\eta}$, v_0 is an assigned reference velocity and H is the externally impressed uniform magnetic field, η is the magnetic viscosity and μ the magnetic permeability.

To the Eqs. (1) we add the initial conditions

$$\boldsymbol{u}(x, y, z, 0) = \boldsymbol{u}_0(x, y, z), \qquad \boldsymbol{h}(x, y, z, 0) = \boldsymbol{h}_0(x, y, z) \quad \text{on } z = \pm 1$$

and the boundary conditions

$$\boldsymbol{u} = 0, \qquad \boldsymbol{h} = 0 \quad \text{on } z = \pm 1, \quad \forall t > 0 \tag{2}$$

in the case of rigid boundaries, and

$$\partial_z u = 0, \qquad \partial_z v = 0, \qquad w = 0 \quad \text{on } z = \pm 1, \quad \forall t > 0 \tag{3}$$

in the case of stress-free boundaries.

We assume that the perturbations are x, y periodic with respect to a rectangle $\mathcal{P} = [-\frac{\pi}{a_x}, \frac{\pi}{a_x}] \times [-\frac{\pi}{a_y}, \frac{\pi}{a_y}]$ with wave numbers a_x and a_y in the x and y directions, respectively.

To exclude the rigid motions of the system in the case of stress-free boundaries we have the additional conditions

$$\int_\Omega u\,dx\,dy\,dz = \int_\Omega v\,dx\,dy\,dz = 0 \quad \text{with } \Omega = \mathcal{P} \times [-1, 1].$$

In order to eliminate the term $\nabla\lambda$ in Eqs. (1) and to obtain variables more appropriate for the definition of Lyapunov function, the following poloidal–toroidal decomposition is applied to the solenoidal periodic fields $\boldsymbol{u}$ and $\boldsymbol{h}$ [8,13–15]:

$$\begin{aligned}&\boldsymbol{u} = \nabla\times(\nabla\varphi\times\boldsymbol{k}) + \nabla\psi\times\boldsymbol{k} + \boldsymbol{f} = \boldsymbol{\delta}\varphi + \boldsymbol{\varepsilon}\psi + \boldsymbol{f}\\ &\boldsymbol{h} = \nabla\times(\nabla\varphi^{(m)}\times\boldsymbol{k}) + \nabla\psi^{(m)}\times\boldsymbol{k} + \boldsymbol{f}^{(m)} = \boldsymbol{\delta}\varphi^{(m)} + \boldsymbol{\varepsilon}\psi^{(m)} + \boldsymbol{f}^{(m)}.\end{aligned}$$

where $\boldsymbol{\delta}\cdot = (\partial_{xz}\cdot, \partial_{yz}\cdot, -\Delta_2\cdot)$, $\boldsymbol{\varepsilon}\cdot = (\partial_y\cdot, -\partial_x\cdot, 0)$, $\Delta_2 = \partial_x^2 + \partial_y^2$, $\boldsymbol{f} = (f_1, f_2, f_3)$, $\boldsymbol{f}^{(m)} = (f_1^{(m)}, f_2^{(m)}, f_3^{(m)})$. The functions φ, ψ, $\varphi^{(m)}$, and $\psi^{(m)}$ are uniquely determined if we require them to be periodic with respect to $\mathcal{P}$ and to have vanishing mean value over $\mathcal{P}$. The first part of the decomposition of $\boldsymbol{u}$ and $\boldsymbol{h}$ is called the poloidal part and the second one the toroidal