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The influence of a nonlinear memory on the damped wave equation

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ABSTRACT

In this paper we study the influence of a nonlinear memory

$$F(t,u)=\int_0^t (t-s)^{-\gamma}\ |\,u(s,x)\,|^p\ ds\,,\quad \gamma\in(0,1)\,,$$

on the global existence of small data solutions to

$$u_{tt}-\triangle u+u_t=F(t,u)\,,\qquad u(0,x)=u_0(x)\,,\quad u_t(0,x)=u_1(x)\,,$$

in space dimension $1\leq n\leq 5$. We prove the global existence for $p>\bar{p}(n,\gamma)$, where $\bar{p}(n,\gamma)$ is the *critical exponent*, i.e. no global weak solution exists for $1<p\leq\bar{p}(n,\gamma)$ for suitable, arbitrarily small, data.

To prove our result, we consider small data in some energy space $H^k\times H^{k-1}$, where $k\geq 1$, with additional L^1 regularity. We also discuss what happens if this latter assumption is dropped, i.e. data are only assumed to be small in $H^k\times H^{k-1}$, for some $k\geq 1$.

1. Introduction

In this paper, we study the global existence of small data solutions to

$$\begin{cases} u_{tt}-\triangle u+u_t=F(t,u),\\ u(0,x)=u_0(x),\\ u_t(0,x)=u_1(x),\end{cases}\tag{1}$$

where

$$F(t,u):=\int_0^t (t-s)^{-\gamma}\,|u(s,\cdot)|^p\,ds,\tag{2}$$

for some $\gamma\in(0,1)$ and $p>1$, represents a nonlinear memory. We also derive decay estimates for the solution to (1).

The function $\Gamma(1-\gamma)\,F(t,u)$, where Γ is the Euler Gamma function, is the Riemann–Liouville integral of $|u(t,\cdot)|^p$ with starting point 0; hence

$$\lim_{\gamma\to 1}\Gamma(1-\gamma)\,F(t,u)=|u(t,\cdot)|^p\quad\text{a.e.}$$

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Therefore, it is reasonable to expect relations with the case of a power nonlinearity $F(u) = |u|^p$, as $\gamma \to 1$. On the other hand, the solution to the Cauchy problem for the linear damped wave equation

$$\begin{cases} u_{tt} - \triangle u + u_t = 0, \\ u(0,x) = u_0(x), \\ u_t(0,x) = u_1(x), \end{cases} \tag{3}$$

i.e. (1) with $F \equiv 0$, behaves asymptotically like the solution to the Cauchy problem for the linear heat equation (see, for instance, [1–3])

$$\begin{cases} v_t - \triangle v = 0, \\ v(0,x) = v_0(x), \end{cases} \tag{4}$$

if one takes $v_0 = u_0 + u_1$ in (4). Thus, it is also reasonable to expect that the Cauchy problem (1) is related to the Cauchy problem for the heat equation with nonlinear memory

$$\begin{cases} v_t - \triangle v = F(t,v), \\ v(0,x) = v_0(x) \geq 0, \end{cases} \tag{5}$$

where

$$F(t,v) := \int_0^t (t-s)^{-\gamma}\, v(s,\cdot)^p\, ds. \tag{6}$$

T. Cazenave, F. Dickstein and B. Weissler proved [4] that the *critical exponent* for (5) is

$$\bar{p}(n,\gamma) := \max\{p_\gamma(n), \gamma^{-1}\}, \tag{7}$$

where

$$p_\gamma(n) := 1 + \frac{2(2-\gamma)}{[n-2(1-\gamma)]_+}. \tag{8}$$

In general, by *critical exponent* $\bar{p} = \bar{p}(n,\gamma)$ for (1) or (5), in this paper we mean that

(a) if $p > \bar{p}$ (or, possibly, $p \in (\bar{p}, \tilde{p}]$, for some $\tilde{p} > \bar{p}$), then there exist global-in-time small data solutions to (1) or (5), for a suitable choice of data and solution spaces;
(b) if $1 < p \leq \bar{p}$, there exist arbitrarily small initial data, such that there exists no global-in-time weak solution to (1) or (5).

Back in 1966, H. Fujita [5] proved that the *critical exponent* for the classical semilinear heat equation, i.e. (5) with $F(v) = v^p$, is $1+2/n$. In [6], G. Todorova and B. Yordanov applied linear $(L^1 \cap L^2) - L^2$ estimates and they proved that the *critical exponent* for small data global solutions to (1) with $F(u) = |u|^p$ remains the Fujita exponent $1 + 2/n$ (the nonexistence result in the critical case $p = 1 + 2/n$ was indeed derived by Qi. S. Zhang [7]). Assumptions on the initial data were later relaxed by R. Ikehata and his collaborators [8–10].

Let us come back and focus on our problem (1). We may explicitly compute the exponent $\bar{p}(n,\gamma)$ which appears in (7):

- in space dimension $n = 1$, it holds

$$\bar{p}(1,\gamma) = p_\gamma(1) = \begin{cases} 1 + \dfrac{2(2-\gamma)}{2\gamma-1} = \dfrac{3}{2\gamma-1} & \text{if } \gamma \in (1/2, 1), \\ \infty & \text{if } \gamma \in (0, 1/2]; \end{cases}$$

- in space dimension $n = 2$, it holds

$$\bar{p}(2,\gamma) = p_\gamma(2) = 1 + \frac{(2-\gamma)}{\gamma} = 2\gamma^{-1},$$

  for any $\gamma \in (0,1)$;
- in space dimension $n \geq 3$, it holds

$$\bar{p}(n,\gamma) = \begin{cases} p_\gamma(n) & \text{if } \gamma \in [(n-2)/n, 1), \\ \gamma^{-1} & \text{if } \gamma \in (0, (n-2)/n]; \end{cases} \tag{9}$$

- the function $\bar{p}(n,\gamma)$ is nonincreasing with respect to n and γ;
- it holds $\bar{p}(n,\gamma) \to 1 + 2/n$, for any $n \geq 1$, as $\gamma \to 1$.

In [11], A. Fino proved blow-up in finite time in two cases:

(i) if $p \in (1, p_\gamma]$, when $\bar{p}(n,\gamma) = p_\gamma$, that is, for any $\gamma \in (0,1)$ if $n = 1, 2$ or for any $\gamma \in [(n-2)/n, 1)$ if $n \geq 3$;
(ii) if $p \in (1, n/(n-2)]$, when $\bar{p}(n,\gamma) = \gamma^{-1}$, that is, $n \geq 3$ and $\gamma \in (0, (n-2)/n]$.