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Markus–Yamabe conjecture for non-autonomous dynamical systems

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ABSTRACT

The aim of this paper is the study of the problem of global asymptotic stability of trivial solutions of non-autonomous dynamical systems (both with continuous and discrete time). We study this problem in the framework of general non-autonomous dynamical systems (cocycles). In particular, we present some new results for a non-autonomous version of the Markus–Yamabe conjecture.

1. Introduction

1.1. Markus–Yamabe conjecture (MYC) [1]

Consider the differential equation

$$u' = f(u) \tag{1}$$

and suppose that the Jacobian $f'(u)$ of f has only eigenvalues with negative real part for all u. The *Markus–Yamabe conjecture* is that if $f(0) = 0$, then 0 is a globally asymptotically stable solution for (1).

It is easy to prove **MYC** for $n = 1$. In the two-dimensional case the affirmative answer to **MYC** was obtained in the works [2–4] (see also the references therein). In the work [5] (see also [6,7] and the references therein) is given a polynomial counterexample to the Markus–Yamabe conjecture. If $n > 2$ there are also some additional conditions forcing the Markus–Yamabe conjecture. For example if $f'(u)$ is negative definite for all $u \in \mathbb{R}^n$ the conjecture was proved in [8,9] (see also [10,11,1]). For triangular systems **MYC** was proved in [1].

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1.2. The discrete Markus–Yamabe conjecture (DMYC) [12,13]

Let f be a C^1 mapping from $\mathbb{R}^n$ into itself such that $f(0) = 0$ and for all $u \in \mathbb{R}^n$, $f'(u)$ has all its eigenvalues with modulus less than one. Then 0 is a globally asymptotically stable solution of the difference equation

$$u(n+1) = f(u(n)). \tag{2}$$

In his book [14] J. P. LaSalle proves the **DMYC** for $n = 1$. The discrete Markus–Yamabe conjecture is true only for planar maps (see [12] and also the references therein) and the answer to the question is yes only in the case of planar polynomial maps. The authors [12] prove that the **DMYC** is true for triangular maps defined on $\mathbb{R}^n$ and for polynomial maps defined on $\mathbb{R}^2$. In the works [15,16] the **DMYC** is proved for gradient maps.

1.3. Belitskii–Lyubich conjecture [17]

Let $\mathfrak{B}$ be a Banach space, $\Omega \subset \mathfrak{B}$ an open subset and $f : \Omega \mapsto \mathfrak{B}$ be compact and continuously differentiable in Ω. Suppose D is a nonempty bounded convex open subset of X such that $f(\overline{D}) \subset \overline{D} \subset \Omega$ and $\sup_{x\in\overline{D}} r(f'(x)) < 1$ ($r(A)$ is the spectral radius of linear bounded operator A). Then the discrete dynamical system $(\overline{D}, f)$, generated by positive powers of $f : \overline{D} \mapsto \overline{D}$, admits a unique globally asymptotically stable fixed point.

The aim of this paper is the study the problem of global asymptotic stability of trivial solutions of non-autonomous dynamical systems (both with continuous and discrete time). We study this problem in the framework of general non-autonomous dynamical systems (cocycles).

The idea of applying methods of the theory of dynamical systems to the study of non-autonomous differential equations is not new. It has been successfully applied to the resolution of different problems in the theory of linear and non-linear non-autonomous differential equations for more than forty years. First this approach to non-autonomous differential equations was introduced in the works of L. G. Deyseach and G. R. Sell [18], R. K. Miller [19], V. M. Millionshchikov [20–22], G. Seifert [23], G. R. Sell [24–26], B. A. Shcherbakov [27,28], later in the works of I. U. Bronstein [29,30], R. A. Johnson [31,32], B. M. Levitan and V. V. Zhikov [33], Sacker R. J. [34,35], Sacker R. J and Sell G. R. [36–44], G. R.Sell, W. Shen and Y. Yi [45], B. A. Shcherbakov [46,47], V. V. Zhikov [48–50] and many other authors. This approach consists of naturally linking with equation

$$x' = f(t, x) \tag{3}$$

a pair of dynamical systems and a homomorphism of the first onto the second. In one dynamical system is put the information about the right hand side of Eq. (3) and in the other about the solutions of Eq. (3).

This paper is organized as follows.

In Section 2 we give some notions and facts from the theory of global attractors of dynamical systems which we use in our paper.

Section 3 is dedicated to the study of non-autonomous dynamical systems with convergence. We present some important tests of convergence (Theorems 3.6, 3.7 and 3.10) of non-autonomous dynamical systems with minimal base.

In Section 4 we study the Markus–Yamabe problem for non-autonomous systems. In this section we prove the necessary and sufficient conditions of global asymptotic stability the trivial section of non-autonomous dynamical systems with continuous or discrete time (Theorem 4.4 — the main result of paper). We apply this result to the different classes of non-autonomous evolution equations (finite-dimensional systems, gradient systems, and triangular systems).

We give in Section 5 some new results concerning the discrete Markus–Yamabe problem for non-autonomous systems. In particular we present the affirmative answer to the **DMYC** for non-autonomous contractive, triangular and potential (gradient) maps.

2. Compact global attractors of dynamical systems

Let X be a topological space, $\mathbb{R}$ ($\mathbb{Z}$) be a group of real (integer) numbers, $\mathbb{R}_+$ ($\mathbb{Z}_+$) be a semi-group of the nonnegative real (integer) numbers, $\mathbb{S}$ be one of the two sets $\mathbb{R}$ or $\mathbb{Z}$ and $\mathbb{T} \subseteq \mathbb{S}$ ($\mathbb{S}_+ \subseteq \mathbb{T}$) be a sub-semigroup of additive group $\mathbb{S}$.

Definition 2.1. Triplet $(X, \mathbb{T}, \pi)$, where $\pi : \mathbb{T} \times X \to X$ is a continuous mapping satisfying the following conditions:

$$\pi(0, x) = x; \tag{4}$$

$$\pi(s, \pi(t, x)) = \pi(s + t, x); \tag{5}$$

is called a dynamical system. If $\mathbb{T} = \mathbb{R}$ ($\mathbb{R}_+$) or $\mathbb{Z}$ ($\mathbb{Z}_+$), then the dynamical system $(X, \mathbb{T}, \pi)$ is called a group (semi-group). In the case, when $\mathbb{T} = \mathbb{R}_+$ or $\mathbb{R}$ the dynamical system $(X, \mathbb{T}, \pi)$ is called a flow, but if $\mathbb{T} \subseteq \mathbb{Z}$, then $(X, \mathbb{T}, \pi)$ is called a cascade (discrete flow).

Sometimes, briefly, we will write xt instead of $\pi(t, x)$.

Below X will be a complete metric space with metric ρ.