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# Diversities, hyperconvexity and fixed points

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ABSTRACT

Diversities have been recently introduced as a generalization of metrics for which a rich tight span theory could be stated. In this work we take up a number of questions about hyperconvexity, diversities and fixed points of nonexpansive mappings. Most of these questions are motivated by the study of the connection between a hyperconvex diversity and its induced metric space for which we provide some answers. Examples are given, for instance, showing that such a metric space need not be hyperconvex but still we prove, as our main result, that they enjoy the fixed point property for nonexpansive mappings provided the diversity is bounded and that this boundedness condition cannot be transferred from the diversity to the induced metric space.



## 1. Introduction

A general theory on *diversities* has recently been proposed by D. Bryant and P. Tupper in [1]. The authors introduce diversities in this work as a sort of multi-way metric which inherits its name after some special appearances in work on phylogenetic and ecological diversities [2–5]. In [1] the authors develop a theory of Tight Span Diversities parallel to the theory of Tight Span for metric spaces independently given by A. Dress [6] and J.R. Isbell [7].

As explained by A. Dress, K.T. Huber and V. Moulton in [8], perhaps the first paper that studied metric spaces as such was the work by J. Isbell [7] on metric tight spans. In this work, J. Isbell finds a natural metric envelope with minimal and uniqueness properties for any given metric space. This metric envelope is named *the hyperconvex hull* by J. Isbell, later rediscovered by A. Dress as *the metric tight span*, who provides a construction for the unique (up to isometries) minimal hyperconvex metric space where a given metric space may be isometrically embedded. Hyperconvex metric spaces had been introduced some years earlier by N. Aronszajn and P. Panitchpakdi in [9] as metric spaces which are absolute nonexpansive retracts. Since then a lot has been written on hyperconvex metric spaces, the reader may find a gentle introduction to most of this information in the recent surveys [10,11] where hyperconvexity and its connections to existence of fixed points for nonexpansive mappings are explained. These surveys do not deal however with the connection of metric tight spans with phylogenetic problems. For this the reader may consult the delightful exposition on this particular point given in [1].

Motivated by the big impact of tight spans in phylogenetic problems and given that there were some particular natural examples of objects which could be understood as generalized metrics after the name of diversities, D. Bryant and P. Tupper [1] took up the problem of developing a theory of tight spans for diversities. This project first needed to introduce diversities as a general object which contained the already known examples as particular cases. Then a whole new theory of

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hyperconvexity for diversities needed to be created. Bryant and Tupper considered these questions by providing us not only with a natural theory of hyperconvex diversities but also showing that a beautifully parallel theory of tight spans existed in this new context. After developing this theory, Bryant and Tupper illustrated their approach in the final sections of [1] for the cases of the so-called diameter and phylogenetic diversities. As a result, these last sections establish very powerful relations between metric tight spans and diversity tight spans of the same metric space when these particular diversities were taken into consideration.

The work that we present here has been directly motivated by Bryant and Tupper's seminal paper on diversities [1] and references to it will be given throughout our work. Our aim is to find out which general connections may be found between a hyperconvex diversity and its induced metric space. In particular we wonder about the existence of fixed points for nonexpansive self-mappings on such an induced metric space. On the way to giving answers to this problem we will need to show new properties and provide examples regarding diversities and induced metric spaces. The work is organized as follows: in Section 2 we review the main facts and definitions from [1] which are relevant to our discussion as well as the main facts on hyperconvex metric spaces which can be found in a more detailed way in any of [10,11]. Section 2 closes with a new result on the problem of extending nonexpansive mappings from an induced metric space to the diversity. In Section 3 we consider general hyperconvex diversities and study which properties the induced metric space inherits. We prove that this metric space need not be hyperconvex itself and give a sufficient condition that guarantees that this metric space is hyperconvex. As particular examples, we show that both the diameter and the phylogenetic diversities satisfy this condition with respect to their natural induced metrics. As our main result in this section we show that if the diversity is hyperconvex then the induced metric space need not have the fixed point property for nonexpansive mapping even if this metric space is bounded. In Section 4 we remove the boundedness condition from the induced metric space to the diversity to show that, in this case, the induced metric space actually has the fixed point property for nonexpansive mappings. We close the work with a positive result on nonempty intersection of decreasing families of hyperconvex and bounded diversities in the spirit of the one given by J.P. Baillon for hyperconvex metric spaces in [12].

## 2. Preliminaries

We begin with metric and hyperconvex metric spaces. Let $(X, d)$ be a metric space, then $\bar{B}(a, r)$ will stand for the closed ball of center $a \in X$ and radius $r \geq 0$. The *Chebyshev radius* of a set $A \subseteq X$ with respect to $x \in X$ will be, as usual, given by

$$r_x(A) = \sup_{a \in A} d(x, a).$$

A subset $A$ of a metric space is said to be admissible if it can be written as an intersection of closed balls. In many aspects admissible subsets of a metric space are a counterpart for convex subset of a linear space. In fact, any subset $A$ of a metric space has an admissible, or ball, hull given by:

$$B(A) = \bigcap\{B : B \text{ is a closed ball containing } A\}.$$

Notice that a subset of a metric space is admissible if and only if $A = B(A)$. Admissible subsets enjoy a number of general properties, the interested reader may check [10] for more details on this, however we will only need the following representation of admissible sets which is immediate to show. If $A$ is an admissible subset of $X$ then

$$A = \bigcap_{x \in X} \bar{B}(x, r_x(A)).$$

**Definition 2.1.** A metric space $M$ is said to be hyperconvex if given any family $\{x_\alpha\}_{\alpha \in \mathcal{A}}$ of points of $M$ and any family $\{r_\alpha\}_{\alpha \in \mathcal{A}}$ of nonnegative numbers satisfying

$$d(x_\alpha, x_\beta) \leq r_\alpha + r_\beta$$

then

$$\bigcap_{\alpha \in \mathcal{A}} B(x_\alpha, r_\alpha) \neq \emptyset.$$

A particular class of hyperconvex spaces is given by $\mathbb{R}$-trees (or real trees) which will be needed in this work.

**Definition 2.2.** An $\mathbb{R}$-tree is a metric space $T$ such that:

(i) there is a unique geodesic segment (denoted by $[x, y]$) joining each pair of points $x, y \in T$;
(ii) if $[y, x] \cap [x, z] = \{x\}$, then $[y, x] \cup [x, z] = [y, z]$.
From (i) and (ii) it is easy to deduce:
(iii) If $p, q, r \in T$, then $[p, q] \cap [p, r] = [p, w]$ for some $w \in T$.

Another notion relevant in the study of hyperconvex spaces, and especially of metric fixed point theory, is that of nonexpansive mapping.