Nonlinear Analysis 95 (2014) 263–280

Contents lists available at ScienceDirect

Nonlinear Analysis

journal homepage: www.elsevier.com/locate/na

Sublinear and superlinear Ambrosetti–Prodi problems for the Dirichlet p-Laplacian

Sergiu Aizicovici [a], Nikolaos S. Papageorgiou [b], Vasile Staicu [c,*]

[a] Department of Mathematics, Ohio University, Athens, OH 45701, USA

[b] Department of Mathematics, National Technical University, Zografou Campus, Athens 15780, Greece

[c] CIDMA, Department of Mathematics, University of Aveiro, Campus Universitário de Santiago, 3810-193 Aveiro, Portugal

ARTICLE INFO

Article history:
Received 6 May 2013
Accepted 28 August 2013
Communicated by Enzo Mitidieri

MSC:
35J20
35J60
35J92

Keywords:
Ambrosetti–Prodi problem
Sublinear and superlinear reaction
Upper and lower solutions
Nonlinear regularity
Critical groups

ABSTRACT

We deal with an Ambrosetti–Prodi problem driven by the p-Laplace differential operator, with a "crossing" reaction which can be sublinear or superlinear (in the positive direction). Using variational methods based on the critical point theory, together with upper–lower solutions, truncation and comparison techniques and critical groups, we show the existence of a unique critical parameter value λ^* such that for $\lambda < \lambda^*$ there are at least two nontrivial solutions, for $\lambda = \lambda^*$ there is at least one nontrivial solution, and for $\lambda > \lambda^*$ no solutions exist. We extend several recent results on this problem.

1. Introduction

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with a C^2-boundary $\partial\Omega$. In this paper, we deal with the following parametric nonlinear Dirichlet problem

$$-\triangle_p u(z) = f(z, u(z)) + \lambda w(z) \quad \text{in } \Omega, \ u|_{\partial\Omega} = 0. \qquad (P_\lambda)$$

Here $\triangle_p$ denotes the p-Laplace differential operator defined by

$$\triangle_p u = div\left(\|Du\|_{\mathbb{R}^N}^{p-2} Du\right), \quad \text{for all } u \in W_0^{1,p}(\Omega), \ 1 < p < \infty.$$

Also, $f(z, x)$ is a Carathéodory perturbation (i.e., for all $x \in \mathbb{R}$, $z \to f(z, x)$ is measurable and for a.a. $z \in \Omega$, $x \to f(z, x)$ is continuous), $\lambda \in \mathbb{R}$ is a parameter and $w \in C(\Omega) \cap L^\infty(\Omega)$ with $w(z) > 0$ for all $z \in \Omega$. We assume that for a.a. $z \in \Omega$, $f(z, .)$ exhibits asymmetric growth near $+\infty$ and near $-\infty$. More precisely, we assume that the quotient $\frac{f(z,x)}{|x|^{p-2}x}$ asymptotically at $-\infty$ remains below the first eigenvalue $\widehat{\lambda}_1$ of $\left(-\triangle_p, W_0^{1,p}(\Omega)\right)$, allowing for partial interaction (nonuniform nonresonance), while asymptotically at $+\infty$ it is above $\widehat{\lambda} > 0$, again with partial interaction possible. In fact, we deal with both

* Corresponding author. Tel.: +351 915433870; fax: +351 234 370066.
E-mail addresses: aizicovs@ohio.edu (S. Aizicovici), npapg@math.ntua.gr (N.S. Papageorgiou), vasile@ua.pt (V. Staicu).

0362-546X/$ – see front matter
http://dx.doi.org/10.1016/j.na.2013.08.026

the $(p-1)$-sublinear case (i.e., $\limsup_{x\to+\infty}\frac{f(z,x)}{x^{p-1}}$ is finite) and the $(p-1)$-superlinear case (i.e., $\lim_{x\to+\infty}\frac{f(z,x)}{x^{p-1}}=+\infty$). For the latter case, we do not employ the usual in such instances (unilateral) Ambrosetti–Rabinowitz condition (the AR-condition for short). Instead, we employ a weaker condition which also incorporates in our framework superlinear perturbations with "slower" growth near $+\infty$. Problems with a reaction exhibiting such an asymmetric growth are known in the literature as "Ambrosetti–Prodi problems". Their study was initiated with the semilinear work (i.e., $p=2$) of Ambrosetti–Prodi [1]. Since then there have been several extensions and variants of the semilinear Ambrosetti–Prodi problem. In this direction, we mention the papers of Berger–Podolak [2], de Figueiredo [3], Hess [4] (the sublinear case) and de Figueiredo–Solimini [5] (the superlinear case). In the aforementioned works it is proved that for such problems, one can find a critical parameter value λ^* such that for all $\lambda<\lambda^*$, the problem has at least two solutions, for $\lambda=\lambda^*$ there exists at least one solution, and for $\lambda>\lambda^*$ the problem has no solutions.

The nonlinear problem (i.e., $p\neq 2$), was studied only recently starting with the work of Mawhin [6] on periodic equations driven by the scalar p-Laplacian. The first papers that focus on partial differential equations involving the p-Laplacian are those of Arcoya–Ruiz [7] and Koizumi–Schmitt [8], both dealing with the $(p-1)$-sublinear case. The $(p-1)$-superlinear case was investigated in the recent works of Arias–Cuesta [9] and Miotto [10]. We should also mention the very recent paper of de Paiva–Montenegro [11] which studies the $(p-1)$-sublinear Neumann Ambrosetti–Prodi problem. In all the aforementioned nonlinear works, it is proved that there exist two critical parameter values $-\infty<\lambda_*\le\lambda^*<+\infty$ such that:

- for all $\lambda<\lambda_*$, problem (P_λ) has at least two solutions;
- for $\lambda\in[\lambda_*,\lambda^*]$, problem (P_λ) has at least one solution;
- for all $\lambda>\lambda^*$, problem (P_λ) has no solutions.

The question of whether $\lambda_*=\lambda^*$ or not was left open in general. Some partial answers with additional restrictions on the data of the problem can be found in Arcoya–Ruiz [7] (see Theorem 4.4, sublinear case) and in Arias–Cuesta [9] (Theorem 5.1, superlinear case). We mention that in all of the above nonlinear works, the approach is topological, based on the Leray–Schauder degree.

In this paper, we use a perturbation $f(z,x)$ which satisfies weaker regularity conditions than in [7–10]. By introducing a condition concerning the behavior of $x\to f(z,x)$ near zero, we are able to prove the exact analog of the "semilinear result". So, we prove the existence of one critical parameter $\lambda^*>0$ (i.e., $\lambda_*=\lambda^*>0$) such that

- for all $\lambda<\lambda^*$, problem (P_λ) has at least two nontrivial solutions;
- for all $\lambda=\lambda^*$, problem (P_λ) has at least one nontrivial solution;
- for all $\lambda>\lambda^*$, problem (P_λ) has no solutions.

Note that here, by virtue of the extra condition on $f(z,.)$ near zero, we are able to show that the solutions produced are nontrivial. Moreover, our approach provides a unified treatment of both the sublinear and superlinear cases. Our tools are variational, based on the critical point theory, coupled with truncation and comparison techniques and with the use of Morse theory (critical groups). In the next section, for easy reference, we recall some of the mathematical tools which will be used in the sequel.

2. Mathematical background

Throughout this paper, by $\|.\|_p$ we denote the norm of $L^p(\Omega)$ or $L^p\left(\Omega,\mathbb{R}^N\right)$, and by $\|.\|$ we denote the norm of the Sobolev space $W_0^{1,p}(\Omega)$. By virtue of the Poincaré inequality we have

$$\|u\|=\|Du\|_p\quad\text{for all } u\in W_0^{1,p}(\Omega).$$

By $\|.\|$ we will also denote the $\mathbb{R}^N$-norm. No confusion is possible, since it will be clear from the context which norm is used.

For every $x\in\mathbb{R}$, we set $x^\pm=\max\{\pm x,0\}$ and then for $u\in W_0^{1,p}(\Omega)$, we set

$$u^\pm(.)=u(.)^\pm.$$

It is well known (see for example, Gasinski–Papageorgiou [12], p. 199) that

$$u^\pm\in W_0^{1,p}(\Omega),\qquad |u|=u^++u^-,\qquad u=u^+-u^-.$$

By $|.|_N$ we denote the Lebesgue measure on $\mathbb{R}^N$. Finally, if $g:\Omega\times\mathbb{R}\to\mathbb{R}$ is a measurable function (for example, a Carathéodory function), then we set

$$N_g(u)(.)=g(.,u(.))\quad\text{for all } u\in W_0^{1,p}(\Omega).$$

Let $(X,\|.\|)$ be a Banach space and X^* its topological dual. By $\langle\cdot,\cdot\rangle$ we denote the duality brackets for the pair (X^*,X) and we will use the symbol "$\xrightarrow{w}$" to designate weak convergence.

Let $\varphi\in C^1(X)$. We say that φ satisfies the *Cerami condition* (*C-condition*, for short), if the following is true:

"every sequence $\{x_n\}_{n\ge1}\subset X$ such that $\{\varphi(x_n)\}_{n\ge1}\subset\mathbb{R}$ is bounded and

$$(1+\|x_n\|)\varphi'(x_n)\to 0\quad\text{in } X^* \text{ as } n\to\infty$$

admits a strongly convergent subsequence".