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The regularity of the distance function propagates along minimizing geodesics

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ABSTRACT

We consider the distance function from the boundary of an open bounded set $\Omega \subset \mathbb{R}^n$ associated to a Riemannian metric with $C^{1,1}$ coefficients. We show that the $C^{1,1}$ regularity propagates, towards the boundary $\partial\Omega$, along the distance minimizing geodesics. Hence, we show that the cut-locus is invariant with respect to the generalized gradient flow associated to the distance function and that it has the same homotopy type as Ω.

1. Introduction and statement of the results

Let $\Omega \subset \mathbb{R}^n$ be an open bounded set and let d be the viscosity solution of the equation

$$\begin{cases} \langle A(x)Dd, Dd\rangle = 1 & \text{in } \Omega, \\ d = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

Here $A(\cdot)$ is a $n \times n$ symmetric matrices valued function, Dd is the gradient of d and $\langle\cdot,\cdot\rangle$ the Euclidean scalar product. A continuous function, $d : \Omega \to \mathbb{R}$, is a *viscosity solution*[1] of (1.1) iff for every φ of class C^1 and every $x \in \Omega$ such that $d - \varphi$ has a local minimum at x we have

$$\langle A(x)D\varphi(x), D\varphi(x)\rangle = 1.$$

We assume that

$$A(\cdot) \text{ is positive definite and } A(\cdot) \text{ is of class } C^{1,1}. \tag{1.2}$$

We need the following

Definition 1.1. A continuous function u is semiconcave in Ω if there exists a positive constant C such that $D^2u \leq CI$ in $\mathcal{D}'(\Omega)$. Furthermore, u is locally semiconcave in Ω if u is semiconcave in every $U \subset\subset \Omega$.

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[1] We point out that our definition of viscosity solution is not the usual one for general first order Hamilton–Jacobi equations. On the other hand, it is well-known (see e.g. [1]) that, in the case of Hamiltonians, convex with respect to the gradient, the usual definition is equivalent to ours.

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Remark 1.1. A semiconcave function can be locally written as the sum of a concave function with a $C^{1,1}$ function (in particular, a semiconcave function is locally Lipschitz continuous). Hence, in general, a semiconcave function is not globally differentiable and may have singularities as a concave function.

We recall a regularity result for the viscosity solution of Eq. (1.1) (see [2] for a more general result).

Theorem 1.1. *Under Assumption (1.2) the viscosity solution of (1.1) is semiconcave in Ω.*

We recall that the viscosity solution of Eq. (1.1) is the distance function from the boundary of Ω associated to the Riemannian metric $g_x(\xi,\xi) = \langle A^{-1}(x)\xi,\xi\rangle$ (see e.g. [3]), i.e. defining $\ell(x,z)$ as

$$\inf\{T \geq 0 \mid \exists y \in W^{1,\infty}([0,T];\Omega) \text{ s.t. } y(0)=x,\ y(T)=z,\ \langle A^{-1}(y(t))y'(t),y'(t)\rangle \leq 1\ t \text{ a.e. in } [0,T]\},$$

we have that

$$d(x) = \inf_{z\in\partial\Omega} \ell(x,z).$$

A distance minimizing geodesic starting at the point x is a curve $y \in W^{1,\infty}([0,d(x)];\Omega)$ such that $\langle A^{-1}(y(t))y'(t),y'(t)\rangle \leq 1$, for t a.e. in $[0,d(x)]$, and $y(d(x)) \in \partial\Omega$. In particular, if y is a distance minimizing geodesic starting at the point x we have

$$d(y(t)) = d(x) - t, \quad t \in [0,d(x)],$$

and, one can show that

$$d(y(t)) = \ell(y(t),y(s)) + d(y(s)) \quad 0 \leq t \leq s \leq d(x). \tag{1.3}$$

In this perspective, our regularity assumptions mean that we are studying a Riemannian distance (i.e. $A^{-1}(\cdot)$ is a non degenerate matrix) and in the distance minimizing geodesics there are no interior branching points (this fact is a consequence of the $C^{1,1}$ regularity of $A^{-1}(\cdot)$.) Indeed, using an elementary modification of an example given in [4], for every $\alpha \in]0,1[$, one can construct a metric of class $C^{1,\alpha}$ such that the local uniqueness of geodesics fails, i.e. distance minimizing geodesics may have (interior) branching points. This can be done by taking as $\Omega \subset \mathbb{R}^2$ a neighborhood of the origin and, for every $\alpha \in]0,1[$,

$$g_x(\xi,\xi) = \left(1 + \frac{4}{(1-\alpha)^2}|x_2|^{1+\alpha}\right)(\xi_1^2+\xi_2^2)$$

with $x=(x_1,x_2)$, $\xi=(\xi_1,\xi_2)\in\mathbb{R}^2$. The singular set of d is defined as follows

$$\Sigma = \{x \in \Omega \mid d \text{ is not differentiable at } x\}.$$

We recall that the closure (in Ω) of the singular set Σ is the cut-locus,[2] $C_A(\Omega)$, i.e.

$$C_A(\Omega) = \overline{\Sigma}.$$

We point out that, even if Σ is not too large (i.e. it is countably $\mathcal{H}-(n-1)$ rectifiable) its closure may have positive n dimensional Lebesgue measure. This phenomenon is a consequence of the low regularity setting we are working with. For an example of distance function in a convex set of the plane with boundary of class $C^{1,1}$ such that the closure of Σ is of positive, 2-dimensional Lebesgue measure we refer the reader to the paper [5].

Remark 1.2. From an analytical point of view, the cut-locus is the singular support C^1 of d, i.e. $x \notin C_A(\Omega)$ iff there exists a neighborhood of x, U, such that $d \in C^1(U)$. We recall that a more precise regularity result holds: if d is differentiable in an open set U then $d \in C^{1,1}(U)$ (see e.g. [2]). In particular, we deduce that the C^1 and the $C^{1,1}$ singular supports coincide.

For a comparison between different notions of cut-locus see e.g. [6].

In the next result we collect some known properties of the distance minimizing geodesics.

Proposition 1.1. *We assume Condition (1.2).*

(1) *For every $x \in \Omega$ there exists a minimizing geodesic starting at x. Furthermore, d is differentiable along the distance minimizing geodesics possibly except at the end points and*

$$y'(t) = -A(y(t))Dd(y(t)), \quad t \in]0,d(y(0))[, \tag{1.4}$$

for every distance minimizing geodesic, $y(\cdot)$, starting at the point $y(0)$.

(2) *For every $x \in \Omega \setminus \Sigma$ there exists a unique distance minimizing geodesic starting at x, $y(\cdot;x) : [0,d(x)[\longrightarrow \Omega$.*

(3) *Let $x_n \in \Omega$ be a sequence of points such that $\lim_{n\to\infty} x_n = x \in \Omega\setminus\Sigma$. Let $y(\cdot;x_n)$ be a sequence of distance minimizing geodesics. Then there exists $\lim_{n\to\infty} y(\cdot;x_n)$ and it is the distance minimizing geodesic starting at the point x.*

(4) *For every $x,z \in \Omega$ let $y(\cdot;x)$ and $y(\cdot;z)$ be two distance minimizing geodesics starting at x and z respectively. Then*

$$\{y(s;x) \mid s \in]0,d(x)[\} \cap \{y(s;z) \mid s \in]0,d(z)[\} = \emptyset.$$

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[2] It is the closure of the points which are starting points of more than one distance minimizing geodesics with different tangents at the starting point.