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Thresholds for global existence and blow-up in a general class of doubly dispersive nonlocal wave equations

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ABSTRACT

In this paper we study the global existence and blow-up of solutions for a general class of nonlocal nonlinear wave equations with power-type nonlinearities, $u_{tt} - Lu_{xx} = B(-|u|^{p-1}u)_{xx}$, $(p > 1)$, where the nonlocality enters through two pseudo-differential operators L and B. We establish thresholds for global existence versus blow-up using the potential well method which relies essentially on the ideas suggested by Payne and Sattinger. Our results improve the global existence and blow-up results given in the literature for the present class of nonlocal nonlinear wave equations and cover those given for many well-known nonlinear dispersive wave equations such as the so-called double-dispersion equation and the traditional Boussinesq-type equations, as special cases.

1. Introduction

The present paper considers the general class of nonlocal nonlinear wave equations of the form

$$u_{tt} - Lu_{xx} = B(g(u))_{xx}, \tag{1.1}$$

where $u = u(x, t)$ is a real-valued function, $g(u) = -|u|^{p-1}u$ with $p > 1$, and L and B are linear pseudo-differential operators with smooth symbols $l(\xi)$ and $b(\xi)$, respectively, and identifies sharp thresholds of global existence and blow-up for solutions with subcritical initial energy. The important point to notice here is that this study extends the global existence and blow-up results established recently in [1] for (1.1) to the case of $g(u) = -|u|^{p-1}u$ $(p > 1)$, where linear semigroups, the contraction mapping principle and the concavity method of Levine are the main tools for proving the global existence and blow-up results. It is also worth pointing out that the present study uses the potential well method based on the concepts of invariant sets suggested by Payne and Sattinger in [2] as a result of studying energy level sets.

Throughout this paper it is assumed that L and B are elliptic coercive operators. Denoting the orders of L and B by ρ and $-r$, respectively, with $\rho \geq 0$, $r \geq 0$, this requirement is identified with the existence of positive constants c_1, c_2, c_3 and c_4 so that

$$c_1^2(1+\xi^2)^{\rho/2} \leq l(\xi) \leq c_2^2(1+\xi^2)^{\rho/2}, \tag{1.2}$$
$$c_3^2(1+\xi^2)^{-r/2} \leq b(\xi) \leq c_4^2(1+\xi^2)^{-r/2}, \tag{1.3}$$

for all $\xi \in \mathbb{R}$.

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The class of nonlocal nonlinear wave equations characterized by (1.1) has been introduced recently in [1] as a generalization of the so-called double-dispersion equation [3,4]

$$u_{tt} - u_{xx} - \gamma_1 u_{xxtt} + \gamma_2 u_{xxxx} = (g(u))_{xx} \tag{1.4}$$

where $\gamma_1 > 0$ and $\gamma_2 > 0$ are constants, and the terms u_{xxtt} and u_{xxxx} represent dispersive effects. Notice that setting $B = (1-\gamma_1\partial_x^2)^{-1}$ and $L = (1-\gamma_1\partial_x^2)^{-1}(1-\gamma_2\partial_x^2)$ in (1.1) yields (1.4). For other reduction examples of (1.1), including the "good", improved or sixth-order Boussinesq equation [5], the reader is referred to [1]. In (1.1), B is a smoothing operator that smoothes out the nonlinear term and it is the source of one type of dispersion. To see the latter fact, we rewrite (1.1) as $B^{-1}u_{tt} - B^{-1}Lu_{xx} = g(u)_{xx}$. Here, the first and second terms on the left-hand side reflect the two sources of dispersive regularization.

For a general function $g(u)$, the Cauchy problem of (1.1) with the initial data

$$u(x, 0) = u_0(x), \qquad u_t(x, 0) = u_1(x), \quad x \in \mathbb{R} \tag{1.5}$$

has been studied in [1] and some global existence and blow-up results have been established. To make our exposition self-contained we repeat the global existence and blow-up results of [1] without proofs. For that purpose, here are the relevant definitions: $G(u) = \int_0^u g(z)dz$, $\Lambda^{-1}u = \mathcal{F}^{-1}\left(|\xi|^{-1}\mathcal{F}u\right)$ (where $\mathcal{F}$ and $\mathcal{F}^{-1}$ denote the Fourier transform and its inverse, respectively, in the x variable), and

$$\mathcal{E}(t) = \frac{1}{2}\left\|B^{-1/2}\Lambda^{-1}u_t(t)\right\|_{L^2}^2 + \frac{1}{2}\left\|B^{-1/2}L^{1/2}u(t)\right\|_{L^2}^2 + \int_{\mathbb{R}} G(u(t))dx. \tag{1.6}$$

Then the two theorems about global existence and blow-up of solutions are as follows.

Theorem 1.1 (*Theorem 6.4 of [1]*). *Assume that $r + \frac{\rho}{2} \geq 1$, $\frac{r}{2} + \frac{\rho}{2} > \frac{1}{2}$, $s > \frac{1}{2}$, $g \in C^{[s]+1}(\mathbb{R})$, $u_0 \in H^s(\mathbb{R})$, $u_1 \in H^{s-1-\frac{\rho}{2}}(\mathbb{R})$, $G(u_0) \in L^1(\mathbb{R})$ and $G(u) \geq 0$ for all $u \in \mathbb{R}$. Then the Cauchy problem (1.1) and (1.5) has a unique global solution $u \in C([0,\infty), H^s(\mathbb{R})) \cap C^1\left([0,\infty), H^{s-1-\frac{\rho}{2}}(\mathbb{R})\right)$.*

Theorem 1.2 (*Theorem 6.5 of [1]*). *Assume that $B^{-1/2}L^{1/2}u_0 \in L^2(\mathbb{R})$, $B^{-1/2}\Lambda^{-1}u_1 \in L^2(\mathbb{R})$, $G(u_0) \in L^1(\mathbb{R})$. If $\mathcal{E}(0) < 0$ and there is some $\nu > 0$ such that*

$$ug(u) \leq 2(1+2\nu)G(u) \quad \textit{for all } u \in \mathbb{R}, \tag{1.7}$$

then the solution $u(x, t)$ of the Cauchy problem (1.1) and (1.5) blows up in finite time.

The above-given theorems are fundamental for describing the behavior of the solutions in many possible cases of the nonlinear function $g(u)$, but they do not cover the situations addressed in this study for the pure power nonlinearities $g(u) = -|u|^{p-1}u$ with $p > 1$. Since $G(u) = -\frac{1}{p+1}|u|^{p+1} \leq 0$, Theorem 1.1 does not cover this particular form of power nonlinearities. This is one source of the motivation for the present study. Another source of the motivation is the restriction $\mathcal{E}(0) < 0$ in Theorem 1.2. In the case of $g(u) = -|u|^{p-1}u$, the condition (1.7) of Theorem 1.2 holds for $\nu = \frac{p-1}{4}$. However it is unclear how to handle the case $\mathcal{E}(0) > 0$ and whether a solution will exhibit finite time blow-up in such a case. To address these issues we attempt here to characterize the dichotomy between global existence and finite time blow-up in the case of the power nonlinearities.

The aim of this study is twofold: to shed light on the issues raised above for the case of $g(u) = -|u|^{p-1}u$ and to provide sharp thresholds for global existence versus blow-up if the initial energy is strictly below a critical energy constant. The main tool of analysis is the potential well method based on the ideas of Payne and Sattinger [2] (for a detailed description of the potential well method the reader is referred, for instance, to [6–11]). Noting that the potential energy, namely the last two terms in (1.6), consists of two parts, the linear part which generates the dispersive effect of the operator $B^{-1}L$ and the purely nonlinear part. It is well known that the threshold for global existence versus blow-up is determined by the competition between this type of dispersion and nonlinearity. Thus we construct certain best constants that relate these two effects via a minimization problem. Then, a critical energy constant d (called the "depth of the potential well" in [2]) at which the effects due to the linear and nonlinear parts of the potential energy are balanced is obtained by solving the minimization problem defined for the total energy functional. Considering the subcritical case, namely assuming that the total energy is less than d, we define two sets of solutions: Σ_+ and Σ_-. The set Σ_+ corresponds to the case where the linear (dispersive) part dominates the nonlinear part while the set Σ_- corresponds to the opposite case. We prove that the above two sets of solutions are invariant under the flow generated by (1.1). Based on this we establish our global existence and finite time blow-up results. In short, the solution of the Cauchy problem (1.1) and (1.5) exists globally in time if the initial data lies in Σ_+ and it blows up in finite time if the initial data lies in Σ_-. Finally we extend the analysis to the case of an augmented critical energy constant $d(\gamma)$ resulting from an augmented functional involving a parameter γ and we analyze the cases where the parameter-dependent results apply whereas those of the parameter independent case fail.

The paper is organized as follows. In Section 2 we cover some preliminaries containing the local existence theorem of [1] and the energy and momentum conservation laws. In Section 3, we first define a constrained minimization problem for the two functionals related to the linear (dispersive) and nonlinear parts of the potential energy and find the critical energy constant. Then, we define the two invariant sets of solutions and establish the threshold for global existence versus blow-up. In Section 4, we extend our considerations to the case of a parameter-dependent objective functional and conclude the