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Positive solution for a superlinear Kirchhoff type problem with a parameter

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A B S T R A C T

This paper is devoted to a nonlinear Kirchhoff type problem depending on a real function and a nonnegative parameter on a smooth bounded domain of $\mathbb{R}^N$, $N \geq 2$. We show that if the nonlinearity is subcritical and superlinear at zero and infinity, then there exists a positive value such that the Kirchhoff type problem has at least one positive solution for any value of the parameter less than the positive value. The main tools are variational methods, critical point theory and iterative techniques.

1. Introduction

In this paper, we study the existence of a positive solution to the following nonlinear Kirchhoff type problem:

$$-\left(a+\lambda m\left(\int_{\Omega}|\nabla u|^2dx\right)\right)\Delta u=f(u),\quad x\in\Omega,\ u|_{\partial\Omega}=0, \tag{1.1}$$

where Ω is a smooth bounded domain of $\mathbb{R}^N$, $N \geq 2$, a is a positive constant, $\lambda \geq 0$ is a parameter, and $m, f : \mathbb{R}^+ \to \mathbb{R}^+$ are continuous functions.

Problem (1.1) is related to the stationary analog of the Kirchhoff equation

$$u_{tt}-\left(a+b\int_{\Omega}|\nabla u|^2dx\right)\Delta u=f(u)$$

proposed by Kirchhoff [1], which extends the classical D'Alembert's wave equation for free vibrations of elastic strings. Problem (1.1) received much attention only after the work of Lions [2], where a functional analysis approach was proposed to attack it. In recent years, the Kirchhoff type problem on bounded domains or $\mathbb{R}^N$ has been studied by many authors; see for example [3–21] and the references therein. From the original motivation of the Kirchhoff type problem, the assumption about m is that $m(s) = a + bs$. In that case, to obtain the existence of solution by applying the Mountain Pass theorem, the

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authors have to impose a 4-superlinear growth condition on the nonlinearity. To overcome this difficulty, it is common to fix $N = 3$ or make a truncation on the function m.

In [3], Alves, Corrêa and Ma studied the existence of positive solutions to the Kirchhoff type problem

$$-M\left(\int_\Omega |\nabla u|^2 dx\right)\triangle u = f(x,u) \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega, \tag{1.2}$$

where Ω is a smooth bounded domain of $\mathbb{R}^N$, M is a positive function, and $f(x,t)$ has subcritical growth on t. Using truncation arguments and uniform a priori estimates of Gidas and Spruck [22], they obtained that if M does not grow too fast in a suitable interval near zero, then the problem (1.2) has at least one positive solution.

Very recently, Li, Li and Shi [12] discussed the existence of positive solution to the following nonlinear Kirchhoff type problem:

$$\left(a + \lambda\int_{\mathbb{R}^N} |\nabla u|^2 dx + \lambda b\int_{\mathbb{R}^N} u^2 dx\right)(-\triangle u + bu) = f(u), \quad x \in \mathbb{R}^N, \tag{1.3}$$

where $N \geq 3$ and a, b are positive constants, and λ is a nonnegative parameter. If f is subcritical, $\lim_{t\to 0}\frac{f(t)}{t} = 0$ and $\lim_{t\to\infty}\frac{f(t)}{t} = \infty$, they got that there exists $\lambda_0 > 0$ such that for any $\lambda \in [0, \lambda_0)$, (1.3) has at least one positive radial solution using a truncation argument and variational methods.

For any continuous function m, so as to discuss the existence of positive solution of the problem (1.1) with respect to the parameter λ, inspired by the method introduced by De Figueiredo, Girardi and Matzeu in [23], where they considered the solution for semilinear elliptic equation with the nonlinearity depending on the gradient of the solution. In this paper, in a bounded convex domain, using a variational method and iterative technique, we establish an existence criterion of positive solution to the problem (1.1).

The main result of this paper is as follows.

Theorem 1.1. *Assume that Ω is convex and*

(H1) *f is local Lipschitz continuous and there are positive constants c and $p \in (2, 2^*)$ such that $f(t) \leq c(1 + t^{p-1})$ for $t > 0$, where $2^* = \infty$ for $N = 2$, $2^* = \frac{2N}{N-2}$ for $N \geq 3$;*

(H2) $\lim_{t\to 0}\frac{f(t)}{t} = 0$;

(H3) $\lim_{t\to\infty}\frac{f(t)}{t} = \infty$.

Then for any positive continuous function m, there exists $\tilde{\lambda} > 0$ such that for any $\lambda \in [0, \tilde{\lambda})$, the problem (1.1) has at least one positive solution.

Note that in the special case of $\lambda = 0$, Theorem 1.1 has been obtained in [24]. Hence, our result can be regarded as an extension of the case $\lambda = 0$ to the more general case considered in (1.1). Also our method is different from that used in [12], we combine variational methods and iterative technique. We point out that if one thinks of problem (1.1) as a perturbation of the classical problem

$$-a\triangle u = f(u), \qquad x \in \Omega,\ u|_{\partial\Omega} = 0,$$

then the main result is quite natural.

This paper is organized as follows. In Section 2, some preliminary concepts and results are presented. Section 3 is devoted to give the proof of the main result.

2. Preliminaries

Let $H_0^1(\Omega)$ be the usual Sobolev space equipped with the norm $\|u\| = (\int_\Omega |\nabla u|^2 dx)^{\frac{1}{2}}$.

For given $w \in H_0^1(\Omega)$ and $\tau \in [\frac{1}{2}, 1]$, define a functional $I_{w,\tau}$ on $H_0^1(\Omega)$ as

$$I_{w,\tau}(u) = \frac{1}{2}(a + \lambda m(\|w\|^2))\|u\|^2 - \tau\int_\Omega F(u)dx, \quad u \in H_0^1(\Omega)$$

where $F(\xi) = \int_0^\xi f(s)ds$. Clearly, by the assumptions imposed on m and f, we know $I_{w,\tau} \in C^1(H_0^1(\Omega), \mathbb{R})$, and

$$I'_{w,\tau}(u)v = (a + \lambda m(\|w\|^2))\int_\Omega \nabla u \cdot \nabla v dx - \tau\int_\Omega f(u)v dx, \quad u, v \in H_0^1(\Omega).$$

In order to obtain the existence of positive solution, we need the following previous result.