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# Bifurcation of critical periods from the reversible rigidly isochronous centers☆

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ABSTRACT

In this paper we study bifurcation of critical periods by perturbing a nonlinear vector field containing $m$-th degree homogeneous terms with a rigidly isochronous center at the origin. First, we give expressions of period bifurcation functions in the form of integrals, and then study the number of critical periods for cases of $m = 1, 2, 3$ respectively.


## 1. Introduction

Consider a two-dimensional analytic real differential system as follows

$$\dot{x} = P(x, y), \qquad \dot{y} = Q(x, y). \tag{1.1}$$

Suppose that (1.1) has a nondegenerate center at the origin $O$. There are two main problems on the study of the behavior of solutions of (1.1) near the origin. One problem is the so-called isochronicity problem, that is, identifying isochronous center in the family of weak centers of finite order in (1.1). Another interesting problem is to consider the critical points of the period function of system (1.1), called critical periods.

Many authors have studied the period function for planar polynomial vector fields and in particular for quadratic ones. See [1–6]. By [7], there are four families of quadratic centers: Hamiltonian ($Q_3^H$), reversible ($Q_3^R$), codimension four($Q_4$) and generalized Lotka–Volterra ($Q_3^{LV}$). The authors of [1,2] proved that systems ($Q_3^H$) and ($Q_4$) have a monotonic period function respectively. Chicone [8] conjectured that the reversible centers have at most two critical periods.

In recent years more and more attention is paid to the bifurcation of critical periods from isochronous vector fields. See [9–12]. Consider a perturbation of system (1.1)

$$\dot{x} = P(x, y) + \varepsilon P_1(x, y), \qquad \dot{y} = Q(x, y) + \varepsilon Q_1(x, y), \tag{1.2}$$

where $\varepsilon$ is a sufficiently small positive number and $O$ is a center which is isochronous for $\varepsilon = 0$. As in [13], the period function $T(\rho, \varepsilon)$ of system (1.2) can be written in the expansion

$$T(\rho, \varepsilon) = T_0 + \sum_{i=1}^{+\infty} T_i(\rho)\varepsilon^i, \tag{1.3}$$

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where $T_0$ is a constant. In [11], the authors call $T_i(\rho)$ the $i$-th period bifurcation function (PBF) and say that $T(\rho, \varepsilon)$ is vanishing of $k$-th order if $T_1(\rho) \equiv T_2(\rho) \equiv \cdots \equiv T_{k-1}(\rho) \equiv 0$ and $T_k(\rho) \not\equiv 0$. The authors of [13] proved that if $T(\rho, \varepsilon)$ is vanishing of $k$-th order and $\rho^*$ is a simple zero of the $k$-th PBF $T_k$ then there is a unique $\rho^*(\varepsilon)$ which tends to $\rho^*$ as $\varepsilon$ tends to 0 and satisfies $T'(\rho^*(\varepsilon), \varepsilon) = 0$. The number of critical periods is discussed in [13] for perturbations of the linear isochronous vector field, i.e.,

$$\dot{x} = -y + \sum_{i=1}^{m} \varepsilon^i P^{(i)}(x, y), \qquad \dot{y} = x + \sum_{i=1}^{m} \varepsilon^i Q^{(i)}(x, y), \tag{1.4}$$

where $P^{(i)}(x, y)$ and $Q^{(i)}(x, y)$ are polynomials with degree less than or equal to $n$. It was proved that there are planar polynomial centers with at least $2[\frac{n-2}{2}]$ critical periods.

The same problem was also discussed in [9] for perturbations of some nonlinear isochronous vector fields, i.e.,

$$\dot{x} = -y + xy + \varepsilon P(x, y), \qquad \dot{y} = x + y^2 + \varepsilon Q(x, y), \tag{1.5}$$

where $P(x, y)$, $Q(x, y)$ are polynomials of degree $n$ of the form

$$P(x, y) = \sum_{l+m=2}^{n} a_{lm} x^l y^m, \qquad Q(x, y) = \sum_{l+m=2}^{n} b_{lm} x^l y^m$$

such that (1.5) has a center at the origin.

In [9], the authors obtained that if $n = 2$, up to the first order in $\varepsilon$, at most one critical period bifurcates from the periodic orbits of the unperturbed system and this bound is sharp; if $n = 3$, up to the first order in $\varepsilon$, at most two critical periods bifurcate from the periodic orbits of the unperturbed system and this bound is sharp; if $n \geq 4$, up to the first order in $\varepsilon$, at most $4[\frac{n+1}{2}] + 1$ critical periods bifurcate from the periodic orbits of the unperturbed system, where $[x]$ denotes the integral part of the real number $x$.

In this paper, we consider a planar system of the form

$$\begin{cases} \dot{x} = (-y + xF_{m-1}(x, y))(1 + \varepsilon\mu(x, y)), \\ \dot{y} = (x + yF_{m-1}(x, y))(1 + \varepsilon\mu(x, y)), \end{cases} \tag{1.6}$$

where $0 < \varepsilon \ll 1$,

$$\mu(x, y) = \sum_{s+t \leq n} b_{st} x^s y^t, \quad n \geq 2, \tag{1.7}$$

and

$$F_{m-1}(x, y) = \sum_{j=1}^{[\frac{m}{2}]} a_{m+1-2j, 2j-1} x^{m-2j} y^{2j-1}, \quad m \geq 2.$$

It is easy to see that system (1.6) has a center at the origin. We will discuss bifurcation of critical periods of this system. In Section 2, we will give expressions of PBFs in the form of integrals. In Section 3, applying our method we study the number of critical periods of the linear isochronous vector field. In Sections 4 and 5, applying our method we further study the number of critical periods of the quadratic and cubic isochronous vector fields respectively.

## 2. Computation of bifurcation: a fundamental theorem

In this section we first establish a fundamental theorem which gives formulas for $T_1(\rho)$ and $T_2(\rho)$ in (1.3) for system (1.6). Then using the formulas we obtain some new results on the number of critical periods for certain cases in the following sections.

With the polar coordinates $x = r\cos\theta$, $y = r\sin\theta$ system (1.6) can be written in the form

$$\begin{cases} \dot{\theta} = 1 + \varepsilon B_1(r, \theta), \\ \dot{r} = r^m B_0(\theta)(1 + \varepsilon B_1(r, \theta)), \end{cases} \tag{2.1}$$

where

$$B_0(\theta) = \sum_{j=1}^{[\frac{m}{2}]} a_{m+1-2j, 2j-1} \cos^{m+1-2j}\theta \sin^{2j-1}\theta, \tag{2.2}$$

$$B_1(r, \theta) = \sum_{s+t \leq n} b_{st} r^{s+t} \cos^s\theta \sin^t\theta. \tag{2.3}$$

Clearly, $\dot{\theta} \equiv 1$ when $\varepsilon = 0$. This means that the unperturbed system $(1.6)|_{\varepsilon=0}$ has an isochronous center at $O$, called a rigidly or uniformly isochronous center. Therefore, system (1.6) is actually a polynomial perturbation of a nonlinear isochronous center.