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Unbounded holomorphic Fourier multipliers on starlike Lipschitz surfaces and applications to Sobolev spaces

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ABSTRACT

By a generalization of Fueter's result, we establish the correspondence between the convolution operators and the Fourier multipliers on starlike Lipschitz surfaces. As applications, we obtain the Sobolev-boundedness of the Fourier multipliers and prove the equivalence between two classes of Hardy–Sobolev spaces on starlike Lipschitz surfaces.

1. Introduction

Since 1970s, the singular integral operators on Lipschitz curves and surfaces have been studied extensively. In 1977, Calderón [1] first proved the L^2 boundedness of the singular Cauchy integral operators on a Lipschitz curve γ, where the Lipschitz constant is small. Later, Coifman–McIntosh–Meyer [2] eliminated this restriction. We refer the reader to Coifman–Jones–Semmes [3], Coifman–Meyer [4] and Jerison–Kenig [5] for further information.

In higher dimensional spaces, let Σ be a Lipschitz surface given by

$$\Sigma = \left\{ g(x)e_0 + x \in \mathbb{R}^{n+1} : x \in \mathbb{R}^n \right\},$$

where g is a Lipschitz function such that $\|\nabla g\|_\infty \le \tan\omega$, $\omega \in [0, \frac{\pi}{2})$. Li–McIntosh–Semmes [6] embedded $\mathbb{R}^{n+1}$ in Clifford algebra $\mathbb{R}_{(n)}$ with identity e_0 and considered the right monogenic functions ϕ satisfying $|\phi(x)| \le C|x|^{-n}$ on a sector S^0_μ, $\mu > \omega$. C. Li, A. McIntosh and T. Qian proved that the convolution singular integral operator

$$T_{(\phi,\underline{\phi})}u(x) = \lim_{\varepsilon\to 0+} \left\{ \int_{y\in\Sigma, |y-x|\ge\varepsilon} \phi(x-y)n(y)u(y)dS_y + \underline{\phi}(\varepsilon n(x))u(x) \right\} \tag{1.1}$$

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is bounded on $L^p(\Sigma)$, $1 < p < \infty$. Gaudry–Long–Qian [7] gave another proof by a $T(b)$ type theorem and a martingale approach.

It is well-known that there exists a correspondence between the convolution singular integrals and the Fourier multipliers. See E. M. Stein's book [8] for details. On infinite Lipschitz curves γ, by the Fourier transform on the sectors and H^∞-functional calculus, McIntosh–Qian [9,10] established the theory of L^p-Fourier multipliers and proved that the convolution singular integral operators on γ are equivalent to Fourier multipliers associated with Dirac operators. See also Gaudry–Qian–Wang [11] and Qian [12] for the theory of L^p Fourier multipliers on the closed Lipschitz curves.

On a Lipschitz surface Σ, Li–McIntosh–Qian [13] generalized the Fourier transform holomorphically to a function of m complex variables. Let $-iD_\Sigma = \sum_{k=1}^{m} -ie_k D_{k,\Sigma}$, where $D_{k,\Sigma} = (\partial/\partial x_k)|_\Sigma u$, $k = 1, \ldots, m$. Li–McIntosh–Qian [13] proved $T_{(\phi,\underline{\phi})}$ defined by (1.1) can be written as $T_{(\phi,\underline{\phi})} = b(-iD_\Sigma) = b(-iD_{1,\Sigma}, -iD_{2,\Sigma}, \ldots, -iD_{m,\Sigma})$, where b is the Fourier transform of ϕ.

The above theory was further extended to the starlike Lipschitz surfaces by Qian [14,15]. Let $S^c_{\omega,\pm}$ and S^c_ω be the sectors defined in Definition 3.4. The results of T. Qian are restricted to the bounded holomorphic Fourier multipliers b belonging to the following classes

$$H^\infty(S^c_{\omega,\pm}) = \left\{ b : S^c_{\omega,\pm} \to \mathbb{C}, b \text{ is bounded holomorphic on } S^c_\omega \text{ and satisfies } |b(z)| \le C_\mu, z \in S^c_{\mu,\pm}, 0 < \mu < \omega \right\}.$$

A natural question is whether there exists a relation between the convolution operators on Σ and the multipliers b dominated by a polynomial, that is,

$$|b(z)| \le C_\mu |z|^s, \quad z \in S^c_{\mu,\pm},\ 0 < \mu < \omega.$$

If Σ is the Euclidean space or its unit sphere, such multipliers are the classical fractional differential and integral operators and are widely applied to the study of harmonic analysis and partial differential equations.

Li–Leong–Qian [16] obtained the following result. Let $\{P^{(k)}\}$ be the monogenic polynomials defined in Definition 3.1. If $b \in H^s(S^c_\omega)$, then the function

$$\phi(x) = \sum_{k \in \mathbb{Z}\setminus\{0\}} b(k) P^{(k)}(x) \in K^s(H_\omega).$$

See Section 3 for the definitions of $H^s(S^c_\omega)$ and $K^s(H_\omega)$. If $s = 0$, the multipliers b become the bounded Fourier multipliers. Hence the above-mentioned result is a generalization of those of McIntosh–Qian [9] and Qian [14,15].

In this article, we consider the converse of the result obtained in [16]. Further, we establish a correspondence between our Fourier multipliers and the kernels of the convolution operators on starlike Lipschitz surfaces. Precisely, we prove that the following are equivalent:

(i) $\phi(x) = \sum_{k \in \mathbb{Z}\setminus\{0\}} b_k P^{(k)}(x) \in K^s(H_{\omega,\pm})$;
(ii) There exists $b \in H^s(S^c_{\omega,\pm})$ such that $b_k = b(k)$, $k \in \mathbb{Z} \setminus \{0\}$.

For the Fourier multipliers on Lipschitz surfaces, there exists a substantial difficulty. On the unit sphere, the L^2-boundedness of Fourier multipliers can be obtained by the Plancherel theorem directly. In the new context, i.e. the starlike Lipschitz surfaces, the Plancherel theorem does not hold. For $s = 0$ and $n = 3$, Qian [14] obtained the equivalence of (i) and (ii). For the case $s \neq 0$, our method is similar to that of Qian [14,15] but with some necessary modification in technology.

(i)$\Longrightarrow$(ii). This part has been obtained by Li-Leong-Qian [16]. For the cases $s > 0$ and $s < 0$, Li-Leong-Qian [16] used the Kelvin inversion and Fueter's result, respectively. We point out that if the dimension n is odd, we could estimate the kernels of the Fourier multipliers for the cases $s > 0$ and $s < 0$ by the same method. We omit the details. See Theorem 3.9.

(ii)$\Longrightarrow$(i). Given ϕ as above. For $s = 0$, such a relation has been obtained by Qian [14]. However, for the case $s \neq 0$, if we apply the method of [14], we can only get $b \in H^{s+2}(S^c_{\omega,\pm})$ rather than $H^s(S^c_{\omega,\pm})$. See Theorem 3.10. Applying Proposition 3.3, we construct the function b by using z^k. This method also helps us avoid the difficulties occurring in the estimate of $|P^{(z)}(x)|$. See Theorem 3.12.

Remark 1.1. In this paper, we assume that the dimension n is odd. We point out that our method cannot be applied to the Clifford algebras with an even number of generators. In fact, our proof of Theorem 3.12 is based on a generalization of Fueter's result which holds when n is odd. See (6) of Proposition 3.3.

In Section 4, we give two applications of the theory of the Fourier multipliers obtained in Section 3. Let $s \in \mathbb{Z}_+ \cup \{0\}$. In Section 4.1, we introduce the Sobolev spaces $W^{2,s}_{\Gamma_\xi}(\Sigma)$ on the starlike Lipschitz surfaces. We prove that if $b \in H^s(S^c_\omega)$, $s > 0$, $r \in \mathbb{Z}_+ \cup \{0\}$, the multiplier operator M_b is bounded from $W^{2,r+s_1}_{\Gamma_\xi}(\Sigma)$ to $W^{2,r}_{\Gamma_\xi}(\Sigma)$. See Theorem 4.3. This result implies that our Fourier multipliers could exert an influence on the index of the Sobolev spaces associated with Γ_ξ.

In Section 4.2, we establish the equivalence between two classes of Hardy–Sobolev spaces on Σ. We can see that there exist two methods to define the Hardy–Sobolev spaces associated with Γ_ξ. By the Fourier multiplier theory, we prove the two classes of Hardy–Sobolev spaces are equivalent. See Theorem 4.4.

Our paper is organized as follows. In Section 2, we state notations, knowledge and terminology which will be used throughout the paper. In Section 3, we obtain the correspondence between the kernel functions ϕ and the multipliers b by use of Fueter's result. In Section 4, we give two applications of the Fourier multiplier theory on starlike Lipschitz surfaces.