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Normally solvable nonlinear boundary value problems

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ABSTRACT

We investigate nonlinear problems which appear as Euler–Lagrange equations for a variational problem. They include in particular variational boundary value problems for nonlinear elliptic equations studied by F. Browder in the 1960s. We establish a solvability criterion of such problems and elaborate an efficient orthogonal projection method for constructing approximate solutions.

1. Introduction

A single nonlinear partial differential equation is as inexhaustible as the whole mathematics. Nevertheless informal theory is still possible for some general classes of nonlinear partial differential equations. As but one example we mention the theory of boundary value problems for higher order elliptic quasilinear equations of divergence type developed in the 1960s by Browder (see [1] and the references given there).

The boundary value problems of [1] can be specified within Euler–Lagrange equations for variational problems with one unknown function lying in a certain vector space. Thus, nonlinear conditions on the boundary are not permitted by the very setting, which restricts essentially applications of the theory. The boundary value problem for an elliptic quasilinear equation of divergence type with homogeneous Dirichlet conditions on the boundary is a typical example of problems treated by Browder.

The class of boundary value problems we study in the present paper is well motivated by applications to overdetermined elliptic systems. As usual such a system possesses no solution, let alone the boundary value problems for solutions of the system. If the boundary conditions may be satisfied by functions leaving out the system then one looks for a function which fulfils the boundary conditions and minimises the discrepancy in the system. This leads to a variational problem for a functional whose critical points are solutions of the so-called Euler–Lagrange equations. These latter can therefore be thought of as relaxation of the original boundary value problem.

The Euler–Lagrange equations represent a boundary value problem for an elliptic system of partial differential equations. This is what can actually be called the nonlinear Laplacian associated with the original problem. If the differential equations

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of the original system are nonhomogeneous then those of Euler–Lagrange equations are nonlinear, too. Moreover, if the original boundary conditions are of mixed type then the boundary conditions of Euler–Lagrange equations bear essential nonlinearities.

A classical reference on variational calculus still unexcelled is the monograph [2]. Some developments along classical lines are presented in [3]. We also mention the paper [4] where a variational approach was developed to study the Cauchy problem for nonlinear elliptic equations with data on a part of the boundary.

The relaxation of boundary value problems to Euler–Lagrange equations has evident advantages. If the original problem possesses a solution then this is among the solutions of Euler–Lagrange equations. Variational problems are amenable to efficient numerical methods, just recall the classical Ritz method [5]. The Euler–Lagrange equations are endowed with weak formulations by the very nature. As but one typical feature of these equations we mention that they are of generic position, i.e., the number of equations just amounts to the number of unknown functions.

By the above, the Euler–Lagrange equations constitute a broad class of nonlinear boundary value problems whose study is well motivated both by internal applications in mathematics and by natural sciences. With this as our main purpose we elaborate in the present paper a variational approach to boundary value problems for systems of nonlinear partial differential equations. To clarify the approach we restrict our attention to first order systems and minimise the discrepancy in the L^2-norm. Hence, we are looking for a function u of Sobolev space $H^1 = W^{1,2}$ which minimises the discrepancy in the system and satisfies an affine equation $Bu = u_0$ at the boundary. The critical points of the discrepancy functional are characterised as weak solutions of a boundary value problem for a system of second order nonlinear differential equations. Apart from the condition $Bu = u_0$ this problem includes an additional nonlinear equation at the boundary. The function u is required to satisfy the relaxed system in the sense of distributions in the interior of the domain. The interpretation of the additional boundary condition is more sophisticated, however, it is understood in a precise weak sense anyway. The ellipticity appears quite naturally in the study.

In Section 2 we discuss the relaxation of the initial problem for a nonlinear ordinary differential equation on a finite interval. We are looking for a global solution of the problem and so minimise the discrepancy in a class of functions satisfying the initial condition. By this example we demonstrate how the so-called Dirichlet to Neumann operator appears to describe the set of all initial data for which the original problem has a solution. In Section 3 we study boundary value problems for a nonlinear elliptic system of first order partial differential equations. The boundary conditions are assumed to be of the form $Bu = u_0$, where B is a right invertible matrix of smooth functions and u_0 a given function on the boundary, cf. [6]. We look for a function which minimises the discrepancy of the system under precisely keeping boundary conditions. We compute the Euler–Lagrange equations of the variational problem. They constitute a boundary value problem for solutions of a second order elliptic system satisfying both $Bu = u_0$ and a suitable trace of the original system on the boundary. The problem is of stable character in the sense that the number of equations just amounts to the number of search-for functions. In this way we obtain what is called a relaxation of the original problem. Yet another designation of this boundary value problem is the nonlinear Laplacian of the original problem, for the way it shows up looks like that of the classical Laplace operator. In Section 4 we examine the algebraic structure of Euler–Lagrange equations and trace out techniques to construct a weak solution. In Section 5 we study the generalised Laplacian of boundary value problems for overdetermined elliptic systems of linear partial differential equations of higher order. If the discrepancy of the system is evaluated in the Hilbert space L^2 in the domain, then the Euler–Lagrange equations represent a boundary value problem for the generalised Laplacian of the original system in the domain with linear boundary conditions. A direct application of Green formula shows that any sufficiently smooth solution of the Euler–Lagrange equations is actually a solution of the original problem. This raises the problem of regularity of weak solutions of Euler–Lagrange equations, as but one example we remind of the $\bar{\partial}$-Neumann problem in complex analysis which initiated ample investigations of subelliptic operators. In fact the situation is much the same for nonlinear boundary value problems. Using generalised Laplacians allows one to reduce nonlinear boundary value problems to a nonlinear integro-differential problem on the boundary. The Dirichlet to Neumann operator treated in Section 6 is a key tool of this reduction. In Section 7 we develop a theory of nonlinear mappings of Banach spaces modelled on elliptic nonlinear boundary value problems investigated in the previous sections. They can be referred to as nonlinear Fredholm mappings and their use goes far beyond boundary value problems, see for instance [7–9]. Finally, in Section 8 we study in detail the Cauchy problem for the derivative operator with data at a boundary piece. This problem is overdetermined, and so we consider a suitable variational relaxation of the problem and prove that it has a unique solution. This allows one to explicitly construct a nonlinear Dirichlet to Neumann operator.

2. A leading example

Let $\mathcal{X} = [a, b]$ be a bounded interval in $\mathbb{R}$ and f a continuous function on $[a, b] \times \mathbb{R}$ with real values. We assume that $f(x, u)$ satisfies a Lipschitz condition in u uniformly in $x \in [a, b]$. Consider the problem of finding a function $u \in H^1[a, b]$ satisfying

$$\begin{cases} u'(x) = f(x, u) & \text{for } x \in (a, b), \\ u(a) = u_0, \end{cases} \tag{2.1}$$

where u_0 is a given number.