Nonlinear Analysis 95 (2014) 543–552

Contents lists available at ScienceDirect

Nonlinear Analysis

journal homepage: www.elsevier.com/locate/na

The obstacle problem for Hessian equations on Riemannian manifolds

Heming Jiao *, Yong Wang

Department of Mathematics, Harbin Institute of Technology, Harbin, 150001, China

ARTICLE INFO

Article history:
Received 12 April 2013
Accepted 7 October 2013
Communicated by Enzo Mitidieri

Keywords:
Obstacle problem
Fully nonlinear equations
Viscosity solution
Regularity

ABSTRACT

In this paper, we consider the obstacle problem for a class of fully nonlinear equations on Riemannian manifolds. The $C^{1,1}$ regularity of the greatest viscosity solutions is established.

1. Introduction

Let (M^n, g) be a complete Riemannian manifold with metric g and $\Omega \subset M$ a bounded domain with C^4 boundary $\partial\Omega$ in M. In this paper we consider the obstacle problem

$$\begin{cases} f(\lambda[\kappa u g + \nabla^2 u]) \geq \psi(x, \nabla u, u) & \text{in } \Omega, \\ u \leq h & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega, \\ u \text{ is admissible in } \Omega, \end{cases} \tag{1.1}$$

where $\kappa \leq 0$ is a constant, f is a symmetric function of $\lambda \in \mathbb{R}^n$, $\nabla^2 u$ denotes the Hessian of a function u on M, for a $(0, 2)$ tensor X on M, $\lambda(X)$ denotes the eigenvalues of X with respect to the metric g.

The function $f \in C^2(\Gamma) \cap C^0(\overline{\Gamma})$ is assumed to satisfy the elliptic condition:

$$f_i \equiv \frac{\partial f}{\partial \lambda_i} > 0 \quad \text{in } \Gamma,\ 1 \leq i \leq n, \tag{1.2}$$

where Γ is an open, convex, symmetric cone $\Gamma \subset \mathbb{R}^n$ with vertex at the origin and $\Gamma^+ \subseteq \Gamma \neq \mathbb{R}^n$ where

$$\Gamma^+ \equiv \{\lambda \in \mathbb{R}^n : \text{ each component } \lambda_i > 0\}.$$

An upper semicontinuous function $u : \Omega \to \mathbb{R}$ is *admissible* if $\lambda(\kappa u(x) g + X) \in \overline{\Gamma}$ for all $x \in \Omega$ and $(p, X) \in J^{2,+}_{\Omega} u(x)$, where

$$J^{2,+}_{\Omega} u(x) = \{(\nabla\phi(x), \nabla^2\phi(x)) : \phi \text{ is } C^2 \text{ and } u - \phi \text{ has a local maximum at } x\}.$$

* Corresponding author. Tel.: +86 13895731375.
E-mail addresses: jiaoheming@163.com (H. Jiao), mathwy@hit.edu.cn (Y. Wang).

0362-546X/$ – see front matter
http://dx.doi.org/10.1016/j.na.2013.10.004

We refer the reader of interest to [1–4] to the concept of k-convex and k-admissible functions where the authors treated the weak solutions to k-Hessian equations.

Note that $u \in C^2(\Omega)$ is admissible if and only if $\lambda[\kappa ug + \nabla^2 u](x) \in \overline{\Gamma}$ for each $x \in \Omega$.

Our interest to study the obstacle problem for Hessian equations on Riemannian manifolds is from their applications to differential geometry such as finding the greatest hypersurface with an obstacle whose Weingarten curvature is bounded below by a nonnegative function as $f = S_k^{1/k}$ (see (1.11)). The case that $f = S_n^{1/n}$ and $M = \mathbb{R}^n$, i.e. the obstacle problems for Monge–Ampère type equations on $\mathbb{R}^n$, are studied by many people. For example, in [5], Xiong and Bao have treated the case that $\kappa = 0$, Lee [6] considered similar problem when $\psi \equiv 1$, $\varphi \equiv 0$ and Ω is a strictly convex domain in $\mathbb{R}^n$ and in [7], Caffarelli and McCann studied the free boundary problem related to optimal transportation problem. Another interesting case is the Hessian quotient equations, i.e. $f = S_{k,l}^{1/(k-l)} \equiv (S_k/S_l)^{1/(k-l)}$, $1 \le l < k \le n$ (see [8]).

Let $\mathcal{A} \equiv \{v : v \text{ is a viscosity solution of (1.1)}\}$ and

$$u(x) \equiv \sup_{v\in\mathcal{A}} v(x). \tag{1.3}$$

We shall construct some smooth and nonsmooth viscosity solutions to (1.1) (see Theorem 1.3 below) for some special cases. As is well known, finding a (viscosity) solution to (1.1) or a (viscosity) subsolution to (1.4) is crucial to using Perron's method and proving the existence of solutions. So it is worth to study in depth in general though it is a difficult job.

In this paper, we assume that $\mathcal{A}$ is not empty. We can show that u is still in the class $\mathcal{A}$ under the elliptic condition (1.2). In this paper we are focused on the regularity of u.

As in [5], we call $\psi(x, p, z)$ has *fine property*, if comparison holds for equation

$$f(\lambda(\kappa ug + \nabla^2 u)) = \psi(x, \nabla u, u) \quad \text{in } \Omega, \tag{1.4}$$

that is, let u (resp. v) be a viscosity subsolution (resp. viscosity supersolution) to (1.4) and $u \le v$ on ∂G, then

$$u \le v \quad \text{in } G,$$

where $G \subset \Omega$ is an arbitrary domain.

Many functions have the fine property, see Section 3 of [9] and Section 2 of [10].

In particular, we can prove the following.

Theorem 1.1. *Suppose f satisfies (1.2) and $\psi(x, p, z) \in C(T^*M \times \mathbb{R}) \ge 0$. If $h \ge \varphi \in C(\partial\Omega)$ and h is upper semicontinuous on $\overline{\Omega}$, Then u defined in (1.3) still belongs to $\mathcal{A}$.*

If, in addition, ψ has fine property and there exists a function $\underline{u} \in \mathcal{A}$ such that $\underline{u}_* = \varphi$ on $\partial\Omega$, where $\underline{u}_*$ is the lower semicontinuous envelope of $\underline{u}$ (see Definition 2.2), then $u \in C(\overline{\Omega})$ and it is the unique (viscosity) solution of

$$\begin{cases} \max\{u - h, -(f(\lambda[\kappa ug + \nabla^2 u]) - \psi(x, \nabla u, u))\} = 0 & \text{in } \Omega, \\ u \ge \underline{u} & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega, \\ u \text{ is admissible in } \Omega. \end{cases} \tag{1.5}$$

In order to obtain more regularity, we need more assumptions. We assume f satisfies (see [11,12])

$$f \text{ is a concave function in } \Gamma, \tag{1.6}$$

$$\sum_i f_i(\lambda)\lambda_i \ge 0 \quad \text{for } \lambda \in \Gamma, \tag{1.7}$$

$$f_j \ge \nu_0 \left(1 + \sum_i f_i\right) \quad \text{for any } \lambda \in \Gamma \text{ with } \lambda_j < 0, \tag{1.8}$$

$$f > 0 \quad \text{in } \Gamma, \qquad f = 0 \quad \text{on } \partial\Gamma, \tag{1.9}$$

and

$$(f_1 \cdots f_n)^{\frac{1}{n}} \ge \mu_0 \text{ in } \{\lambda \in \Gamma : \psi_0 \le f(\lambda) \le \psi_1\}, \quad \text{for any } \psi_1 > \psi_0 > 0. \tag{1.10}$$

Our typical example for such f is $S_k^{1/k}$, where S_k is the k-th elementary symmetric function

$$S_k(\lambda) = \sum_{i_1<\cdots<i_k} \lambda_{i_1} \cdots \lambda_{i_k}, \quad 1 \le k \le n, \tag{1.11}$$

defined on the cone

$$\Gamma_k = \{\lambda \in \mathbb{R}^n : S_j(\lambda) > 0,\ j = 1, \ldots, k\}.$$