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The global existence and the life span of smooth solutions to a class of complex conservation laws

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ABSTRACT

In this paper, we consider the Cauchy problem for a class of complex conservation laws, which is introduced by P.D. Lax and essentially a class of quasilinear hyperbolic systems in two space dimensions. For a kind of flux functions, we prove the global existence of smooth solutions to the Cauchy problem with small initial data. On the other hand, for different kinds of flux functions, we give some estimates on the lower bound of the life span of smooth solutions.

1. Introduction

Hyperbolic systems may describe many physical phenomena, a typical example is the shock wave. Since the nonlinear phenomenon associated with hyperbolic systems, particularly in several space dimensions, is too complex, up to now only a few results on the global existence or blowup phenomenon of smooth solutions have been known. For the Cauchy problem of general symmetric hyperbolic systems, the local existence theorem on smooth solutions has been proved by Kato [1], Majda [2] and Lax [3]. For the global existence and the blowup phenomenon of smooth solutions, we refer to Grassin [4], Sideris [5] and Xin [6] for Euler or Navier–Stokes equations, and Sideris [7] for nonlinear hyperbolic equations.

In this paper, we consider the Cauchy problem for a class of complex conservation laws, which is introduced by Lax [8] and is essentially a class of quasilinear hyperbolic systems in two space dimensions. For completion of the paper, we recall some basic facts about this kind of complex conservation laws.

1.1. Basic equations

Lax [8] introduces and investigates the following complex conservation law

$$\frac{\partial \overline{U}}{\partial t} + \frac{\partial (F(U))}{\partial z} = 0, \tag{1.1}$$

where $U = U(t,z)$ is the complex unknown function, $z = x + iy$ stands for the complex variable, $\overline{U}$ is the conjugate of U, while $F = F(U)$ denotes the complex flux function. Let

$$U = u(t,x,y) + iv(t,x,y) \tag{1.2}$$

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and

$$F'(U) = 2[a(u, v) + ib(u, v)], \tag{1.3}$$

where $u = u(t, x, y)$ and $v = v(t, x, y)$ stand for the real part and the imaginary part of $U = U(t, x, y)$, respectively, while $a = a(u, v)$ and $b = b(u, v)$ denote the real part and the imaginary part of $\frac{1}{2}F'(U)$, respectively. It follows from (1.2) and (1.3) that

$$\frac{\partial F(U)}{\partial z} = au_x - bv_x + bu_y + av_y + i(bu_x + av_x - au_y + bv_y). \tag{1.4}$$

Thus, (1.1) can be rewritten as

$$U_t + A(U)U_x + B(U)U_y = 0, \tag{1.5}$$

where

$$U = \begin{pmatrix} u \\ v \end{pmatrix}, \qquad A(U) = \begin{pmatrix} a(u, v) & -b(u, v) \\ -b(u, v) & -a(u, v) \end{pmatrix} \quad \text{and} \quad B(U) = \begin{pmatrix} b(u, v) & a(u, v) \\ a(u, v) & -b(u, v) \end{pmatrix}. \tag{1.6}$$

(1.6) shows that (1.5) is a symmetric hyperbolic system in two space dimensions with two unknown functions $u(t, x, y)$ and $v(t, x, y)$.

More precisely, we rewrite (1.5) as

$$\begin{cases} \partial_t u + a(u, v)\partial_x u - b(u, v)\partial_x v + b(u, v)\partial_y u + a(u, v)\partial_y v = 0, \\ \partial_t v - b(u, v)\partial_x u - a(u, v)\partial_x v + a(u, v)\partial_y u - b(u, v)\partial_y v = 0 \end{cases} \tag{1.7}$$

and consider the Cauchy problem for system (1.7) with the following initial data

$$t = 0 : u(0, x, y) = u_0(x, y) = \epsilon f(x, y), \qquad v(0, x, y) = v_0(x, y) = \epsilon g(x, y), \tag{1.8}$$

where ϵ is a small positive parameter, and f and g are two smooth given functions with compact support. We shall investigate the global existence and the life span of smooth solutions of the Cauchy problem (1.7)–(1.8).

1.2. Main theorems

Throughout the paper, we always assume that

$$a^2 + b^2 = c^2, \tag{1.9}$$

where c is a positive constant. Without loss of generality, we may suppose that $c \equiv 1$, and then there exists a function $\varphi = \varphi(u, v)$ such that

$$a(u, v) = \cos\varphi(u, v), \qquad b(u, v) = \sin\varphi(u, v). \tag{1.10}$$

Suppose furthermore that in a neighborhood of $\lambda \triangleq (u, v) = 0$, say, for $|\lambda| \leqslant 1$, the function $\varphi(\lambda)$ is a suitably smooth function satisfying

$$\varphi(\lambda) = O\left(|\lambda|^{1+n}\right), \tag{1.11}$$

where n is a non-negative integer.

Remark 1.1. Assumption (1.9) ensures that system (1.7) is strictly hyperbolic and linearly degenerate in the sense of Lax [8], which can be seen by Lemma 3.1 in Section 3.

The main results in the present paper are given by the following theorems.

Theorem 1.1. *Assume (1.9)–(1.10) with $n \geqslant 2$, then there exists a positive constant $\epsilon_0 > 0$ such that, for any $\epsilon \in [0, \epsilon_0]$, the Cauchy problem (1.7)–(1.8) has a unique global smooth solution on $[0, \infty) \times \mathbb{R}^2$.*

Define the life span T_ϵ is the supremum of $T > 0$ such that the Cauchy problem (1.7)–(1.8) has a smooth solution on $[0, T] \times \mathbb{R}^2$. Then we have the following.

Theorem 1.2. *Assume (1.9)–(1.10) with $n = 0, 1$, then the life span T_ϵ of the smooth solution of the Cauchy problem (1.7)–(1.8) satisfies*

$$T_\epsilon \geqslant \frac{C}{\epsilon^{2(n+1)}}, \tag{1.12}$$

where C is a positive constant independent of ϵ.