Nonlinear Analysis 95 (2014) 580–591

Contents lists available at ScienceDirect

Nonlinear Analysis

journal homepage: www.elsevier.com/locate/na

On the stability of global solutions to the 3D Boussinesq system

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ARTICLE INFO

Article history:
Received 22 May 2013
Accepted 2 October 2013
Communicated by Enzo Mitidieri

MSC:
35Q30
76D05
76E20

Keywords:
Boussinesq system
Global solution
Stability

ABSTRACT

In this paper, we investigate the stability to any given global smooth solutions of the 3D Boussinesq system. Assuming that $(\overline{\theta}, \overline{u})$ is a global solution (called reference solution later) with initial data $(\overline{\theta}_0, \overline{u}_0)$, we prove that a small perturbation of $(\overline{\theta}_0, \overline{u}_0)$ still generates a global smooth solution to the Boussinesq system, and this solution keeps close to the reference solution.

1. Introduction

The following Boussinesq system describes the influence of the convection (or convection–diffusion) phenomenon in a viscous or inviscid fluid

$$\begin{cases} \partial_t \theta + u \cdot \nabla \theta - \mu \Delta \theta = 0, \\ \partial_t u + u \cdot \nabla u - \nu \Delta u + \nabla \Pi = \theta e_3 \quad \text{with } e_3 = (0, 0, 1), \\ \operatorname{div} u = 0 \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R}^3. \end{cases} \tag{1.1}$$

This system is used as a toy model for geophysical fluids in the regimes where rotation and stratification play an important role (see for example Pedlosky [1]). In the above, $u = u(x, t)$ denotes the velocity vector-field and $\theta = \theta(x, t)$ is a scalar quantity such as the concentration of a chemical substance or the temperature variation in a gravity field, in which case θe_3 represents the buoyancy force. The nonnegative parameters μ and ν denote the molecular diffusion and the viscosity, respectively. Without loss of generality we take the molecular diffusion μ and the viscosity ν of the fluid equal to one.

The main result of this paper concerns with the stability of large global smooth solutions for the Boussinesq system (1.1) under suitable small perturbations. As for the homogeneous Navier–Stokes equations, the first result in this direction is due to [2], in which the authors proved the stability of mildly decaying global strong solutions in three space dimensions under H^1-framework and assuming the condition

$$\int_0^\infty \|\nabla u(., t)\|_{L^2}^4 dt < \infty.$$

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0362-546X/$ – see front matter
http://dx.doi.org/10.1016/j.na.2013.10.011

After that, new stability conditions was obtained in [3] for global strong solutions in $C([0,\infty);L^3)$ (see also [4]). Recently, Gui and Zhang [5] studied the stability to the global large solutions of 3-D incompressible Navier–Stokes equations in the anisotropic Sobolev spaces. In [6] Abidi, Gui and Zhang investigated the large-time decay and stability to any given global smooth solutions of the 3D incompressible inhomogeneous Navier–Stokes equations.

The main motivation of studying the stability of global strong solutions for Navier–Stokes equations is that: when the global existence problem of smooth solutions is not completely solved, but one has global existence of solutions presenting certain type of symmetry (see for instance [7–9]), this kind of stability becomes particularly interesting because it provides a non-symmetric large strong global solution as a small perturbation of a symmetric one. This is the case of the Navier–Stokes equations that have global large strong solutions with axial, rotational, and helical symmetry, see for example [6,5,2]. Recently, there are some results concerning global well-posedness for the Boussinesq system (1.1) with large axisymmetric data in dimensional three, see [10,11]. It is natural to consider the stability of global strong solutions for the Boussinesq system (1.1).

When the solution is small, there are some stability results. Ferreira and Villamizar-Roa in [12] gave a class of stable steady solutions in weak-L^p spaces, in the sense that they only assume that the stable steady solution belongs to scaling invariant class $L^{(n,\infty)}_{\sigma}\times L^{(n,\infty)}$. The authors in [13] investigated well-posedness of mild solution and existence of self-similar ones in the framework of Morrey space. We point out that these small global solutions in weak-L^p and Morrey spaces may be large (even unbounded) in the classical norms L^2, H^1 and in Besov spaces with positive regularity. More about the stability of small global solutions for the Boussinesq system can be found in [14–16] and the references therein.

Now we state our main results. We start by recalling the basic existence result of weak solutions to the system (1.1), see [17].

Proposition 1.1. *Let $(\theta_0,u_0)\in L^2\times L^2_{\sigma}$. There exists a weak solution (θ,u) of the Boussinesq system (1.1) with data (θ_0,u_0), continuous from $\mathbb{R}^+$ to L^2 with the weak topology, such that for any $T>0$,*

$$\theta\in L^2(0,T;H^1)\cap L^\infty(0,T;L^2_{\sigma}),\qquad u\in L^2(0,T;V)\cap L^\infty(0,T;L^2_{\sigma}).$$

Such a solution satisfies, for all $t\in[0,T]$, the energy inequalities

$$\|\theta(t)\|^2+2\int_0^t\|\nabla\theta(s)\|^2ds\le\|\theta_0\|^2,$$

and

$$\|u(t)\|^2+2\int_0^t\|\nabla u(s)\|^2ds\le C(\|u_0\|^2+t^2\|\theta_0\|^2),$$

for all $t\ge 0$ and some constant $C>0$.

Our first result is the following theorem which concerns the global existence.

Theorem 1.1. *Assume $(\theta_0,u_0)\in H^1\times H^1$.*

(a) *If $\theta_0\in L^1\cap L^1_1$ and $\int\theta_0=0$, there exists an absolute constant $\varepsilon_0>0$ such that if*

$$\|\theta_0\|_1<\varepsilon_0,$$

then the weak solution constructed in Proposition 1.1 is a global one satisfying

$$\theta\in C_b([0,\infty);H^1(\mathbb{R}^3))\cap L^1_{\text{loc}}(\mathbb{R}^+;\dot{B}^2_{2,1}(\mathbb{R}^3)),$$
$$u\in\{C_b([0,\infty);H^1(\mathbb{R}^3))\cap L^1_{\text{loc}}(\mathbb{R}^+;\dot{B}^2_{2,1}(\mathbb{R}^3))\}\cap\left\{C_b\left([0,\infty);B^{\frac12}_{2,1}(\mathbb{R}^3)\right)\cap L^1_{\text{loc}}\left(\mathbb{R}^+;\dot{B}^{\frac52}_{2,1}(\mathbb{R}^3)\right)\right\},$$
$$\nabla\Pi\in L^1_{\text{loc}}\left(\mathbb{R}^+;\dot{B}^{\frac12}_{2,1}(\mathbb{R}^3)\right).$$

Furthermore,

$$\|\nabla\theta(t)\|_2\le C(1+t)^{-\frac74},\quad t>0,$$

for some positive constant C.

(b) *Assume additionally that $\theta_0\in\dot{B}^{-\frac32}_{2,1}(\mathbb{R}^3)$ besides the conditions in* (a)*, we have*

$$\theta\in C_b\left([0,\infty);\dot{B}^{-\frac32}_{2,1}(\mathbb{R}^3)\right)\cap L^1_{\text{loc}}\left(\mathbb{R}^+;\dot{B}^{\frac12}_{2,1}(\mathbb{R}^3)\right)\cap L^1_{\text{loc}}(\mathbb{R}^+;\dot{B}^3_{2,1}(\mathbb{R}^3)).$$