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Blow-up solutions in one-dimensional diffusion models

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ABSTRACT

A sufficient and necessary condition for the existence of blow-up solutions of a certain class of nonlinear Volterra integral equations with kernels arising from various diffusion models and with nonlinearities satisfying the condition $g(0) = 0$ is given in the form of the so-called generalized Osgood condition. Such nonlinearities were not permitted in the original articles of Olmstead and Roberts where these diffusion models were introduced.

1. Introduction

The class of Volterra integral equations of the convolution type

$$u(t) = \int_0^t k(t-s)g(u(s))\mathrm{d}s, \quad t \geq 0, \tag{1}$$

appears in many problems of mathematical physics. One of the richest sources for such Volterra equations is diffusion theory, classical as well as anomalous. For instance, Olmstead and Roberts in a series of articles [1–3] derived several integral Volterra equations of the form (1), where the nonlinearity g satisfies some further requirements (particularly $g(0) \neq 0$), typical for explosion models. With these assumptions, they examined the existence of blow-up solutions of (1) with some special kernels k uniquely determined by the type of the diffusion problem under consideration. Unfortunately, when $g(0) = 0$ their methods cannot be used for studying the blow-up solutions of the equation of type (1). On the other hand, knowledge about the existence of blow-up solutions of Eq. (1) in such a case could be used to improve some results obtained by Olmstead and Roberts (see [4] for the example of the improvement of the estimation of the blow-up time). Motivated by this fact, in this article we find a sufficient and necessary condition for the existence of blow-up solutions of (1) with the kernels coming from Olmstead and Roberts' works and with the nonlinearities satisfying the condition $g(0) = 0$. For this purpose, we combine well-known results (such as a comparison theorem or Mydlarczyk condition [5] for the existence of a blow-up solution of (1) with the kernel $k(t) = t^{\alpha-1}$, $\alpha > 0$) with the new integral sufficient condition for the existence of a blow-up solution of (1) obtained as a consequence of the results from [6].

2. Background information

We consider Eq. (1), where k is a locally integrable function positive almost everywhere in $[0, \infty)$ and $g : [0, \infty) \to [0, \infty)$ is a strictly increasing, absolutely continuous function that satisfies the following conditions:

$$g(0) = 0, \tag{2}$$

$$x/g(x) \to 0 \quad \text{as } x \to 0^+, \tag{3}$$

$$x/g(x) \to 0 \quad \text{as } x \to \infty. \tag{4}$$

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By a nontrivial solution of (1) we mean a continuous function u, which satisfies (1), defined on the maximal interval of its existence $[0, T)$, where $T > 0$, such that $u(0) = 0$ and $u(t) > 0$ for $t \in (0, T)$. If additionally $T < \infty$ and $u(t) \to \infty$ as $t \to T^-$, we call such a nontrivial solution a blow-up solution (with a blow-up time T). In [7] it was shown that the assumptions about g and k that we just adopted imply that Eq. (1) has at most one nontrivial solution u and, moreover, u is then a strictly increasing absolutely continuous function. We also assume that the kernel k satisfies the additional condition

$$\lim_{t\to\infty} K(t) \geq \gamma \max_{t\in(0,\infty)} \frac{t}{g(t)}, \tag{5}$$

where $K(t) := \int_0^t k(s)\mathrm{d}s$ and $\gamma > 1$ is an arbitrary number. It was shown in [8] that condition (5) is the necessary condition for the existence of blow-up solutions of (1). Obviously, K is a strictly increasing continuous function; thus it has an inverse function that we denote by K^{-1}.

3. Formulation of the problem

As we mentioned earlier, equations of the form (1) are quite typical in nonlinear diffusion theory. To demonstrate this, we now give a short description of three one-dimensional models of diffusion considered by Olmstead and Roberts that lead to the Volterra integral equation (1). All these models are originally described by the initial–boundary value problems for PDE but they can be reduced to Eq. (1) via Green's functions. For all integral equations derived in this manner, we not only give the formulae for their kernels, but also give either formulae for respective functions K or, when the elementary formulae for K could not be found, the information about the asymptotic behaviour of these functions. We will see later that such knowledge will be crucial for our further investigation of the existence of possible blow-up solutions of these equations.

Model 1 — *the subdiffusion in the bounded spatial domain* (*see* [3] *for more detailed analysis*)

The problem of thermal behaviour of a subdiffusive medium (for instance some certain porous materials in which microscopic pores are filled with a substance with a conductivity lower than that of basic matrix material) can be described by the equation

$$\frac{\partial}{\partial t}T(x,t) = \frac{\partial^2}{\partial x^2}D_t^{1-\alpha}\left[T(x,t)\right] + \delta(x-a)g(T(x,t)), \quad 0 < x < l,\ t > 0, \tag{6}$$

where parameters a, l satisfy the inequality $0 < a < l < \infty$, with the initial condition

$$T(x,0) = 0, \quad 0 \leq x \leq l, \tag{7}$$

and the Dirichlet boundary conditions

$$T(0,t) = T(l,t) = 0, \quad t > 0. \tag{8}$$

The fractional derivative operator $D_t^{1-\alpha}$ in (6) is given by

$$D_t^{1-\alpha}\left[T(x,t)\right] = \frac{1}{\Gamma(\alpha)}\frac{\partial}{\partial t}\int_0^t (t-\tau)^{\alpha-1}T(x,\tau)d\tau \tag{9}$$

and $0 < \alpha < 1$. Transforming the initial–boundary value problem (6)–(8), we obtain the Volterra integral equation (1) with

$$u(t) \equiv T(a,t)$$

and

$$k(t) = \frac{2}{l}\sum_{n=1}^{\infty}\sin^2\left(\frac{n\pi a}{l}\right)\int_0^{\infty} f_\alpha(z)\exp\left(-\frac{n^2\pi^2}{l^2}t^\alpha z\right)\mathrm{d}z, \tag{10}$$

where

$$f_\alpha(z) = \sum_{j=0}^{\infty}\frac{(-1)^j z^j}{j!\Gamma(1-\alpha-\alpha j)}, \quad z \geq 0. \tag{11}$$

The asymptotic behaviour of k is the following:

$$k(t) \sim \frac{1}{2\Gamma\left(1-\frac{\alpha}{2}\right)}t^{-\frac{\alpha}{2}} \quad \text{as } t \to 0^+, \tag{12}$$

$$k(t) \sim \frac{a(l-a)}{l\Gamma(1-\alpha)}t^{-\alpha} \quad \text{as } t \to \infty, \tag{13}$$

and as a consequence

$$K(t) \sim \frac{1}{2\left(1-\frac{\alpha}{2}\right)\Gamma\left(1-\frac{\alpha}{2}\right)}t^{1-\frac{\alpha}{2}} \quad \text{as } t \to 0^+,$$

$$K(t) \sim \frac{a(l-a)}{l\Gamma(2-\alpha)}t^{1-\alpha} \quad \text{as } t \to \infty.$$