Nonlinear Analysis 95 (2014) 676–690

Contents lists available at ScienceDirect

Nonlinear Analysis

journal homepage: www.elsevier.com/locate/na

On the limit cycles of a class of Kukles type differential systems

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ARTICLE INFO

Article history:
Received 4 June 2013
Accepted 5 October 2013
Communicated by Enzo Mitidieri

MSC:
34C29
34C25
47H11

Keywords:
Limit cycle
Kukles systems
Averaging theory

ABSTRACT

In this paper we study the limit cycles of two families of differential systems in the plane. These systems are obtained by polynomial perturbations with arbitrary degree on the second component of the standard linear center. The classes under consideration are polynomial generalizations of certain canonical form of a Kukles system with an invariant ellipse, previously studied in the literature. We provide, in both cases, an accurate upper bound of the maximum number of limit cycles that the perturbed system can have bifurcating from the periodic orbits of the linear center, using the averaging theory of first, second and third order. These upper bounds are presented in terms of the degree of the respective systems. Moreover, the existence of a weak focus with the highest order is also studied.

1. Introduction and statement of the results

One of the main problems in the qualitative theory of real planar differential systems is the determination of their limit cycles, as defined by Poincaré [1]: existence, number and stability. For instance, the second part of the 16th Hilbert problem [2,3] wants to find an upper bound on the maximum number of limit cycles that a polynomial vector field of a fixed degree can have. We shall consider a very particular case, and we will try to give a partial answer to this problem for the *Kukles type systems:* the polynomial differential systems of the form

$$\dot{x} = -y, \qquad \dot{y} = x + \lambda y + \sum_{d=2}^{n} g_d(x, y), \tag{1.1}$$

where $\cdot = \frac{d}{dt}$, $\lambda \in \mathbb{R}$ and $g_d(x, y) \in \mathbb{R}[x, y]$ is a homogeneous polynomial of degree $d \in \mathbb{Z}^+$.

These systems with $\lambda = 0$ has either a center or a weak focus at the origin. This singular point is a *center*, if there is a neighborhood of the origin in which every orbit except the origin is periodic. Similarly, the origin is a *focus*, if it has a neighborhood in which every orbit spirals towards or away from the origin. Thus, the research of (1.1) is closed related with the classical problem of distinguishing between a center and a focus (the center-focus problem), and it is completely solved only for linear and quadratic systems, and a few particular cases in families of higher degree. To describe the mentioned property of the origin, we choose a one-sided analytic transversal at the origin with a local analytic parameter h and represent the Poincaré return map by an expansion $r(h) = h + \sum_{i=0}^{\infty} v_i h^i$. Consequently, the stability of the origin is clearly given by the sign of the first non-zero v_i (a Lyapunov quantity). If all the v_i are zero then the origin is a center. If the displacement function $\delta(h) = r(h) - h$ is not flat (i.e. there exists i such that its ith derivative $\delta^{(i)}(0) \neq 0$) we have a *weak focus*, that means a focus whose associated eigenvalues are pure imaginary. The origin is a *weak focus of order k* if $v_i = 0$ for each $i \leq 2k$, but

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0362-546X/$ – see front matter
http://dx.doi.org/10.1016/j.na.2013.10.013

$v_{2k+1} \neq 0$. It is well-known that at most k limit cycles can bifurcate from a weak focus of order k under perturbation of the coefficients of $\sum_{d=2}^{n} g_d(x, y)$. However, the major difficulty with the functions v_i's is their high complexity, and to find them explicitly becomes a computational problem. For more details about the definitions and statements of this paragraph see, for instance [4,5].

The research of (1.1) was initiated by Kukles [6], giving necessary and sufficient conditions in order that (1.1) with $n = 3$ has a center at the origin. This cubic system without the term y^3 is the so called *reduced Kukles system*. Under this restriction, the authors of [7] present an exhaustive study of the center conditions given in [6] which are also important in the problem of the full classification of cubic systems with a center. It is also considered in [8], where the authors show that at most five limit cycles bifurcate from the origin and they construct a reduced Kukles systems in which this bound is realized. The authors of [8] also solve the center-focus problem for the reduced Kukles systems. In [9] appears a description of the local bifurcations of critical periods in the neighborhood of a non-degenerate center of the reduced Kukles systems, and the authors describe the isochronous systems. In [10], the authors study the number and distributions of the limit cycles for a family of reduced Kukles system under cubic perturbations; by using techniques of bifurcation theory and qualitative analysis, they describe three different distributions of five limit cycles for the systems considered. In [11], the author proves that some cubic systems of the form (1.1) can have seven limit cycles and solve the center-focus problem for a new family of cubic Kukles systems. In [12], we can find some interesting center characterizations of cubic systems, obtained by using symbolic calculations. In [13] is studied (1.1) with $\lambda = 0$ and $\sum_{d=2}^{n} g_d(x, y) \in \{g_4(x, y), g_5(x, y)\}$, for these homogeneous Kukles systems the author studies the center conditions and the existence of small amplitude limit cycles. In [14] the authors consider a class of cubic systems (1.1) having an invariant parabola, and they describe some parameters for which the invariant parabola coexists with a center. It is complemented in [15] where the authors also present a cubic system of the form (1.1) with an invariant hyperbola coexisting with two limit cycles. In short in all these studies (1.1) is at most a system with $n = 5$.

1.1. The Sáez–Szántó's differential systems

We are particularly interested in studying the maximum number of small amplitude limit cycles of a class of Kukles type systems which can coexist with closed invariant algebraic curves. It is initially studied in [16], where the authors describe a class of quintic systems of the form (1.1) having an invariant conic, and they show the coexistence of small amplitude limit cycles bifurcating from the origin with an algebraic limit cycles given by an invariant ellipse. In the papers [17,18] appear this coexistence in systems of arbitrary degree, as we will describe in the comments. In this way, the present paper is strongly influenced by the methods and ideas from [19–21]. Following [22], we apply the averaging theory in order to study the maximum number of limit cycles which can bifurcate from the linear center, perturbed in a special class of systems. Specifically, we consider the system

$$\begin{aligned} \dot{x} &= -y \\ \dot{y} &= x + (x^2 + y^2) \sum_{\ell \geq 1} \varepsilon^{\ell} \big(q_{\ell}(x, y) - A_{\ell}\big), \end{aligned} \tag{1.2}$$

where $A_\ell > 0$ and the polynomial $q_\ell(x, y)$ has degree $n_\ell - 2 \geq 1$ with $q_\ell(0, 0) = 0$.

Theorem 1.1. *Assume that for $\ell = 1, 2, 3$ the constants $A_\ell > 0$, the polynomials $q_\ell(x, y)$ have degree $n_\ell - 2$ and $q_\ell(0, 0) = 0$. Suppose that $n_\ell \in \{2k_\ell, 2k_\ell - 1\}$ and $k_\ell \geq 2$. Then for $|\varepsilon| \neq 0$, sufficiently small the maximum number of limit cycles of (1.2) bifurcating from the periodic orbits of the linear center $\dot{x} = -y$, $\dot{y} = x$ using the averaging theory*

(a) *of first order is $k_1 - 2$;*
(b) *of second order is* $\max \left\{k_2 - 2; 2\left[\frac{n_1-2}{2}\right] - 2\right\}$;
(c) *of third order is* $\max \left\{k_3 - 2; \left[\frac{n_2-2}{2}\right] - 1\right\}$.

Comments to the first theorem:

- Theorem 1.1 was motivated by the results of [18]. In this paper, E. Sáez and I. Szántó introduce the systems of the form

$$\begin{aligned} \dot{x} &= -y \\ \dot{y} &= x + \varepsilon (x^2 + y^2) \left(\sum_{i=1}^{n_1-2} (q_i x^i + \tilde{q}_i y^i) - A_1 \right), \end{aligned} \tag{1.3}$$

  where $q_i, \tilde{q}_i \in \mathbb{R}$ and $A_1 > 0$. This class is obtained by using the canonical form of Kukles type systems with an invariant ellipse [17].
- In [17], the authors present the mentioned canonical form and discuss the tangential 16th Hilbert problem [23] for those polynomial systems, showing an upper bound for the number of bifurcated limit cycles depending on the degree of the system. More precisely, they perturb a Hamiltonian system and obtain a family of Kukles type systems of degree $n \in \{2k, 2k - 1\}$ with $k \geq 3$ whose number of limit cycles is bounded by $k - 1$, and one of the limit cycles is given by an invariant ellipse.