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A singular function with a non-zero finite derivative on a dense set

Juan Fernández Sánchez [a], Pelegrí Viader [b,*], Jaume Paradís [b], Manuel Díaz Carrillo [c]

[a] Grupo de Investigación de Análisis Matemático, Universidad de Almería, Spain
[b] Departament d'Economia i Empresa, Universitat Pompeu Fabra, Ramon Trias Fargas 25–27, 08005, Barcelona, Spain
[c] Departamento de Análisis Matemático, Universidad de Granada, Spain

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ABSTRACT

The authors had exhibited in a previous paper a continuous strictly increasing singular function from [0, 1] into [0, 1] with a derivative that takes non-zero finite values at the points of an uncountable set. In this article, the construction is improved to encompass a dense set.

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1. Introduction

Since the first singular function from [0, 1] into [0, 1] was exhibited, back in 1884, by Cantor [1–3], astonishing everyone with a continuous non-decreasing function, constant on a set of measure one but growing from 0 to 1, much water has gone under the bridge.

In this paper, by a singular function, we understand a monotone increasing and continuous function whose derivatives vanish a.e.

Two decades later, Minkowski created another singular function, in this case strictly increasing, $?(x)$ [4], with the purpose of showing the countable nature of irrational quadratics.

Other singular functions became familiar at the time of Minkowski's one, most of them of a similar nature but with different origin. We may collect them all under the name of Riesz–Nagy, [5–10]. In some cases, the definition of the function has a geometric origin as the pointwise limit of a sequence of fractal deformations of the identity function from [0, 1] into [0, 1]. This has the advantage of showing the self-replicating character of the limit function easily but hinders the study of the derivative.

A closed analytic expression has been discovered for most of the functions mentioned by translating their definition into the confrontation of two different systems of representation for the real numbers in the domain and the range. It is the distortion caused by the two systems that explains the singularity of the function and offers, at the same time, a powerful instrument to study the properties of the subsets where the derivative vanishes or is infinite. The analysis of this distortion

* Corresponding author.
E-mail addresses: juanfernandez@ual.es (J. Fernández Sánchez), pelegri.viader@upf.edu (P. Viader), jaume.paradis@upf.edu (J. Paradís), madiaz@ugr.es (M. Díaz Carrillo).

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led to the initial proofs of the singularity of these functions, and it contributed to the study of the points where the derivative was 0 or ∞ [11–13]. A special mention must be made of two important results obtained very recently [14,15] regarding the points of differentiability of Minkowski's function.

Other singular functions have appeared in the literature in recent years. Worth mentioning are the Okamoto and Wunsch functions [16] that will be used extensively in the present article. Actually, powerful constructions have been used to develop new singular functions, some of them joining the singular aspect with fractal ideas [17–19].

All these functions, though, presented the same behaviour in terms of the range of their derivatives: zero or infinity. Not a single one presented a non-zero finite derivative at a single point.

In a previous paper in this journal, [20], we exhibited a singular function, $J : [0, 1] \longrightarrow [0, 1]$ with a derivative equal to 1 in an uncountable subset of the unit interval. As we have already mentioned, this behaviour is quite unexpected in a singular function. We claimed there that ours was the first example of this, but we must set the record straight by mentioning that, unnoticed by us, some forty years ago, a paper was published [21] where a singular function – different from the ones aforementioned – had a finite non-zero right-derivative at 0. We discovered that very recently as a colleague sent us the reference.

In this paper we extend our previous result to a set which is uncountable and dense. We begin by using the same line of reasoning we used in our previous contribution. Then, with the help of Fubini's Theorem for the differentiability a.e. of a series of functions we will obtain a non-zero finite derivative on the points of a dense set in [0, 1].

This set will be defined with the help of the ternary numeration system and needs to be defined in a sort of self-replicating way such that all points in it have a "tail" (all ternary digits after a given place) which guarantees that our function is differentiable and we can control its value but at the same time they must have a "head" (any finite number of the initial ternary places) that places the number as closely as we wish from any other real number in [0, 1].

2. Previous results

In the mentioned paper, [20], we introduced some sets and some notation that we will keep.

The ternary system in [0, 1] is used to describe several sets of numbers and to help in our calculations. A number $x = \sum_{i=1}^{\infty} \frac{a_i}{3^i}$, where $a_i \in \{0, 1, 2\}$ will be denoted as $0.a_1a_2a_3\ldots$ or $[a_1, a_2, a_3, \ldots]$. Terminating (or finite) expansions for which $a_i = 0$ after a given place n have an infinite counterpart where a_n is replaced with $a_n - 1$ and $a_i = 2$ from there onwards. This does not raise problems except when we deal with derivatives in which case, as they are countable, we should not worry about them.

Let us recall how to build Cantor's set, C, in [0, 1]. We proceed by stages.

In stage one, we remove from [0, 1] the interval

$$C_{1,1} = (0.1, 0.2)\,,$$

which leaves us with $[0, 0.1] \cup [0.2, 1]$.

The second stage removes the central interval in each of the two extant intervals:

$$C_{2,1} = (0.01, 0.02)\,; \qquad C_{2,2} = (0.21, 0.22)\,,$$

and leaves us with $[0, 0.01] \cup [0.02, 0.1] \cup [0.2, 0.21] \cup [0.22, 1]$ and so on. Each stage consists of the removal of the central part of each of the previous stage remaining intervals.

In this way, stage k removes 2^{k-1} intervals $C_{k,j}$ where

$$C_{k,j} = \left(a_{k,j}, b_{k,j}\right) = (0.a_1a_2\cdots a_{k-1}1, 0.a_1a_2\cdots a_{k-1}2)$$

where $a_1, a_2, \ldots, a_{k-1} \in \{0, 2\}$ and $j = 1, \ldots, 2^{k-1}$.

C_k will denote the union of the removed intervals at stage k: $C_k = \bigcup_{j=1}^{2^{k-1}} C_{k,j}$.

After this process, Cantor's set, C, is the set of real numbers whose ternary expansions can be written without the digit 1 and the interval [0, 1] gets partitioned as

$$[0, 1] = C \cup \left(\bigcup_{k=1}^{\infty} C_k\right). \tag{1}$$

Definition 1. The frequency of occurrence of digit $d \in \{0, 1, 2\}$ in the ternary expansion of a number $x = [a_1, a_2, \ldots]$ is

$$\text{freq}(d) = \lim_{n\to\infty} \frac{\#\{i : a_i = d;\ i = 1, \ldots, n\}}{n}$$

when the limit exists.

Definition 2. Given a number $x = [a_1, a_2, \ldots]$, a "head" of x is any finite initial segment of its digits. The corresponding "tail" of x is the final digits after the head.

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