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Sobolev-like cones of trace-class operators on unbounded domains: Interpolation inequalities and compactness properties

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ABSTRACT

In this paper we extend the compactness properties for trace-class operators obtained by Dolbeault, Felmer and Mayorga-Zambrano to a smooth unbounded domain $\Omega \subseteq \mathbb{R}^d$, $d \geq 3$. We consider V, a non-negative potential on Ω that blows up at infinity, and the normed space $H_V(\Omega) = \left\{u \in H_0^1(\Omega) : \|u\|_V^2 = \int_\Omega (|\nabla u(x)|^2 + |u(x)|^2 V(x))\, dx < \infty\right\}$. A positive self-adjoint trace-class operator R belongs to the Sobolev-like cone $\mathcal{H}_{V,+}^1$ if $(\psi_{i,R})_{\mathbb{N}} \subseteq H_V(\Omega)$ and $\langle\!\langle R \rangle\!\rangle_V = \sum_{i=1}^\infty \nu_{i,R} \|\psi_{i,R}\|_V^2 < \infty$, where $(\nu_{i,R})_{i\in\mathbb{N}}$ is the sequence of occupation numbers of R and $(\psi_{i,R})_{i\in\mathbb{N}} \subseteq L^2(\Omega)$ is a corresponding Hilbertian basis of eigenfunctions. We prove that a sequence in $\mathcal{H}_{V,+}^1$, bounded in energy $\langle\!\langle \cdot \rangle\!\rangle_V$, has a subsequence that converges in trace norm; this is analogous to the classical Sobolev immersion $\mathrm{H}^1(\Omega) \subseteq \mathrm{L}^2(\Omega)$. We prove the existence of lower bounds for nonlinear free energy functionals and, by doing so, we establish Lieb–Thirring type inequalities as well as some Gagliardo–Nirenberg type interpolation inequalities; then our compactness result is applied to minimize nonlinear free energy functionals working on $\mathcal{H}_{V,+}^1$.

1. Introduction

Self-adjoint positive trace-class operators $R : L^2(\Omega) \to L^2(\Omega)$, $\Omega \subseteq \mathbb{R}^d$, appear quite naturally in the Heisenberg picture of quantum mechanics (see e.g. [1]). By the Riesz–Schauder and Hilbert–Schmidt Theorems, there exist a sequence of eigenvalues $(\nu_{i,R})_{i\in\mathbb{N}} \subseteq \mathbb{R}_*^+$ and a Hilbertian basis of eigenfunctions $(\psi_{i,R})_{i\in\mathbb{N}} \subseteq L^2(\Omega)$. Because of their interpretation in physics, an eigenvalue $\nu_{i,R}$ is usually referred to as an *occupation number* and the corresponding eigenfunction $\psi_{i,R}$ is referred to as a *wavefunction*; a *mixed state* is a pair $(\nu_{i,R}, \psi_{i,R})_{i\in\mathbb{N}}$ (see e.g. [2] and [3]).

Throughout this work we shall assume that $\Omega \subseteq \mathbb{R}^d$ is an unbounded domain, $d \geq 3$, and that the operators R are such that the corresponding eigenfunctions belong to the Sobolev space $H_0^1(\Omega)$ and its energy

$$\sum_{i=1}^{\infty} |\nu_{i,R}| \left(\int_\Omega |\nabla \psi_{i,R}(x)|^2 + |\psi_{i,R}(x)|^2 V(x)\, dx \right)$$

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is finite. Here $V : \Omega \to \mathbb{R}$ is a prescribed non-negative locally integrable potential verifying

$$\lim_{|x|\to\infty} \operatorname{ess} V(x) = \infty. \tag{1.1}$$

We denote the set of these operators by $\mathcal{H}_V^1$.

In this paper we extend the results of [4] where Ω was assumed bounded. Our main result (Theorem 4.1) is a compactness property for the Sobolev-like cone

$$\mathcal{H}_{V,+}^1 = \{L \in \mathcal{H}_V^1 / L \geq 0\},$$

that is, a sequence in $\mathcal{H}_{V,+}^1$, bounded in energy $\langle\langle\cdot\rangle\rangle_V$, has a subsequence that converges in trace norm to an operator in $\mathcal{H}_{V,+}^1$. As will be seen, the unboundedness of Ω is compensated by the property (1.1) because it implies the compactness of the immersion $H_V(\Omega) \subseteq L^q(\Omega)$, $q \in [2, 2^*]$ (see Proposition 2.1).

To achieve our goal, we consider a class of nonlinear free energy functionals (sometimes called generalized entropy functionals) like

$$\mathcal{F}_{V,\beta}(R) = \operatorname{Tr}((-\Delta + V)R + \beta(R)), \quad R \in \mathcal{H}_{V,+}^1, \tag{1.2}$$

which has been used for a number of applications concerning partial differential equations (see e.g. [3,5–9] and [10]). We prove the existence of lower bounds for functionals more general than (1.2) and, by doing so, we establish Lieb–Thirring type inequalities as well as some Gagliardo–Nirenberg type interpolation inequalities. That these two kinds of inequalities are related to each other is known; see e.g. [11] and [2].

As a technical condition for proving Theorem 4.1 we shall require (see condition (V_α) in Section 3) the Schrödinger operator $-\Delta + V$ to have its first eigenvalue isolated, $0 < \lambda_{V,1} < \lambda_{V,2} \leq \lambda_{V,3} \leq \cdots$, and it should be verified that $\lim_{i\to\infty} \lambda_{V,i} = \infty$, and also the corresponding sequence of eigenfunctions $(\phi_{V,i})_{i\in\mathbb{N}} \subseteq H_0^1(\Omega) \cap H^2(\Omega)$ has to be a Hilbertian basis of $L^2(\Omega)$.

Our setting could physically correspond to an external potential having a singularity, as is the case for some potentials generated by doping charged impurities in semiconductors. For the relation of the kind of results that we obtain to the estimation of the first eigenvalue of the Schrödinger operator $-\Delta + V$ and the analysis of the stability of repulsive Schrödinger–Poisson systems, we refer the reader to [2].

This paper is organized as follows. In Section 2 we give a short review of definitions and present the Sobolev-like cone $\mathcal{H}_V^1$ together with some of its properties—in particular, a regularity result (Proposition 2.3) for the density functions associated with operators in $\mathcal{H}_{V,+}^1$. In Section 3, a Casimir class of functions is introduced for defining nonlinear free energy functionals; then we prove Lieb–Thirring and Gagliardo–Nirenberg type inequalities: Propositions 3.1–3.3, and Theorem 3.1. Section 4 is dedicated to our main result, Theorem 4.1, which establishes a compactness property that is analogous to the classical Sobolev immersion but at trace-class operator level. Finally, we use this result to minimize nonlinear free energy functionals in Section 5.

2. Definitions and preliminary results

Let $\Omega \subseteq \mathbb{R}^d$ be an unbounded domain, $d \geq 3$, with boundary of class C^1. We denote by $\mathcal{L} = \mathcal{L}(L^2(\Omega))$ the set of bounded linear operators acting on $L^2(\Omega)$. By $\mathcal{I}_\infty = \mathcal{I}_\infty(L^2(\Omega))$ and $\mathcal{S}_\infty = \mathcal{S}_\infty(L^2(\Omega))$ we denote, respectively, the spaces of compact operators and compact self-adjoint operators. We also consider the space of trace-class operators (see e.g. [12])

$$\mathcal{I}_1 = \left\{ R \in \mathcal{L} : \sum_{i=1}^{\infty} |(\psi_i, R\psi_i)_{L^2(\Omega)}| < \infty \right\} \subseteq \mathcal{I}_\infty, \tag{2.1}$$

where $(\psi_i)_{i\in\mathbb{N}}$ is any Hilbertian basis of $L^2(\Omega)$. The trace of an operator $R \in \mathcal{I}_1$ is given by

$$\operatorname{Tr}(R) = \sum_{i=1}^{\infty} (\psi_i, R\psi_i)_{L^2(\Omega)}. \tag{2.2}$$

Due to the Riesz–Schauder and Hilbert–Schmidt Theorems (see e.g. [13]), for a given $R \in \mathcal{S}_\infty$, there exists $(\nu_{i,R})_{i\in\mathbb{N}} \subseteq \mathbb{R}$ and a Hilbertian basis $(\psi_{i,R})_{i\in\mathbb{N}} \subseteq L^2(\Omega)$ such that

$$R\psi_{i,R} = \nu_{i,R}\psi_{i,R}, \quad \text{for all } i \in \mathbb{N}. \tag{2.3}$$

We shall assume that $(|\nu_{i,R}|)_{i\in\mathbb{N}}$ is ordered, that is

$$|\nu_{i,L}| \geq |\nu_{j,L}|, \quad \text{for all } i, j \in \mathbb{N},\ i \leq j;$$

if $\nu_{i,R}$ and $-\nu_{i,R}$ are both eigenvalues, then $-|\nu_{i,R}|$ comes first.

On the space $\mathcal{S}_1 = \mathcal{I}_1 \cap \mathcal{S}_\infty$ the trace norm $\|\cdot\|_1$ is given by

$$\|R\|_1 \equiv \operatorname{Tr}(|R|) = \sum_{i=1}^{\infty} |\nu_{i,R}| < \infty. \tag{2.4}$$

We consider a potential $V : \Omega \to \mathbb{R}$ verifying the conditions

(H1) $V(x) \geq 0$ a.e. $x \in \Omega$,