Nonlinear Analysis 93 (2013) 197–225

Contents lists available at ScienceDirect

Nonlinear Analysis

journal homepage: www.elsevier.com/locate/na

Initial trace of solutions of semilinear heat equations with absorption

Waad Al Sayed[a], Laurent Véron[b,*]

[a] College of Sciences and Humanities, Fahad Bin Sultan University, Tabuk, Saudi Arabia

[b] Department of Mathematics, Université François Rabelais, Tours, France

ARTICLE INFO

Article history:
Received 8 April 2013
Accepted 20 July 2013
Communicated by Enzo Mitidieri

MSC:
35K58
35K15

Keywords:
Heat equation
Singularities
Harnack inequalities
Radon measures
Borel measures
Barrier
Initial trace

ABSTRACT

We study the initial trace problem for positive solutions of semilinear heat equations with strong absorption. We show that in general this initial trace is an outer regular Borel measure. We emphasize in particular the case where u satisfies (E) $\partial_t u - \Delta u + t^\alpha |u|^{q-1}u = 0$, with $q > 1$ and $\alpha > -1$ and prove that in the subcritical case $1 < q < q_{\alpha,N} := 1 + 2(1+\alpha)/N$ the initial trace establishes a one to one correspondence between the set of outer regular Borel measures in $\mathbb{R}^N$ and the set of positive solutions of (E) in $\mathbb{R}^N \times \mathbb{R}_+$.

1. Introduction

In this paper we study the initial trace problem for positive solutions of

$$\partial_t u - \Delta u + g(x,t,u) = 0 \text{ dans } Q_T^\Omega := \Omega \times (0,T) \tag{1.1}$$

where Ω is an open domain in $\mathbb{R}^N$, $g \in C(\Omega \times \mathbb{R}_+ \times \mathbb{R})$ such that $g(x,t,\cdot)$ is nondecreasing $\forall (x,t) \in \Omega \times \mathbb{R}$ and $rg(x,t,r) \geq 0$ for all $(x,t,r) \in \Omega \times \mathbb{R}_+ \times \mathbb{R}$. Our first result establishes the existence of an initial trace.

Theorem A. *Assume g satisfies the above conditions and that Eq. (1.1) possesses a barrier at any $z \in \Omega$. If $u \in C^1(Q_T^\Omega)$ is a positive solution of (1.1), it admits an initial trace which belongs to the class of outer regular positive Borel measures in Ω.*

The barrier assumption will be made precise later on in full generality. It is fulfilled if $g(x,t,r) = h(x)t^\alpha |r|^{q-1}r$ with $\alpha > -1$, $q > 1$ and $h \in L^\infty_{loc}(\Omega)$ satisfies $\inf \operatorname{ess} h > 0$ for any compact subset $K \subset \Omega$, or if g satisfies the Keller–Osserman condition, that is $g(x,t,r) \geq h(r) \geq 0$ where h is nondecreasing and there exists a such that

$$\int_a^\infty \frac{ds}{\sqrt{H(s)}} \quad \text{where } H(s) = \int_0^s h(t)dt. \tag{1.2}$$

* Corresponding author. Tel.: +33 0 247367260; fax: +33 0 247367068.
E-mail addresses: waadalsayed@hotmail.com (W. Al Sayed), veronl@univ-tours.fr, laurent.veron49@gmail.com (L. Véron).

0362-546X/$ – see front matter
http://dx.doi.org/10.1016/j.na.2013.07.032

The initial trace of positive solutions of (1.1) exists in the following sense: *there exist a relatively closed set $\mathcal{S} \subset \Omega$ and a positive Radon measure μ on $\mathcal{R} := \Omega \setminus \mathcal{S}$ with the following properties:*

(i) *for any $x_0 \in \mathcal{S}$ and any $\epsilon > 0$*

$$\lim_{t\to 0} \int_{B_\epsilon(x_0)\cap\Omega} u(x,t)dx = \infty, \tag{1.3}$$

(ii) *for any $\zeta \in C_c(\mathcal{R})$*

$$\lim_{t\to 0} \int_{\Omega} u(x,t)\zeta(x)dx = \int_{\Omega} \zeta d\mu. \tag{1.4}$$

The couple $(\mathcal{S}, \mu)$ is unique and characterizes a unique positive outer regular Borel measure ν on Ω.

A similar notion of boundary trace has been introduced by Marcus and Véron [1] in the study of positive solutions of

$$-\Delta u + g(x,u) = 0 \quad \text{in } \Omega. \tag{1.5}$$

This notion itself has turned out to be a very useful tool for classifying the positive solutions of (1.5).

In the second part we concentrate on the particular case of equation

$$\partial_t u - \Delta u + t^\alpha |u|^{q-1}u = 0 \text{ dans } Q_T^\Omega \tag{1.6}$$

where $T > 0, \alpha > -1$ and $q > 1$. Among the most useful tools, we point out the description of positive solutions with an isolated singularity at $(a, 0)$ for $a \in \Omega$, whenever they exist: they are solutions u of (1.6) in Q_T^Ω, which belong to $C^{2,1}\left(Q_T^\Omega\right) \cap C\left(\Omega \times [0,T) \setminus \{(a,0)\}\right)$ and satisfy

$$u(x,0) = 0 \quad \text{in } \Omega \setminus \{a\}. \tag{1.7}$$

When $\alpha = 0$, Brezis and Friedman prove in [2] that if $B_{2R}(a) \subset \Omega$, then any such solution satisfies

$$u(x,t) \le \frac{C(N,q,R)}{\left(|x-a|^2 + t\right)^{\frac{1}{q-1}}} \quad \forall (x,t) \in B_R(a) \setminus \{0\} \times [0,T]. \tag{1.8}$$

They also prove that if $1 < q < q_N := 1 + \frac{2}{N}$ and $k > 0$ there exist singular solutions with initial data $u(\cdot,0) = k\delta_a$, unique if u vanishes on $\partial\Omega \times [0,T]$. In this range of exponents, Brezis, Peletier, and Terman obtain in [3] the existence and uniqueness of a *very singular solution* of (1.6), always with $\alpha = 0$: it is a positive solution in $Q_\infty := Q_\infty^{\mathbb{R}^N}$ under the form

$$v_0(x,t) = t^{-1/(q-1)} V_0\left(\frac{x}{\sqrt{t}}\right),$$

where $V_0 > 0$ is C^2 and satisfies

$$\begin{aligned} &-\Delta V_0 - \frac{1}{2}\eta.\nabla V_0 - \frac{1}{q-1}V_0 + V_0^q = 0 \quad \text{in } \mathbb{R}^N \\ &\lim_{|\eta|\to\infty} |\eta|^{\frac{2}{q-1}} V_0(\eta) = 0. \end{aligned} \tag{1.9}$$

Actually, Kamin and Peletier [4] show that v_0 is the limit of the solutions u_k of (1.6) in Q_∞ which satisfy $u(\cdot,0) = k\delta_0$. The very singular solution plays a fundamental role in Marcus and Véron's description [5] of the initial trace of positive solutions of (1.6) with $\alpha = 0$. In [6], Marcus and Véron study this equation when $\alpha \ge 0$ and $1 < q < q_{\alpha,N} = 1 + \frac{2(1+\alpha)}{N}$. They obtain the existence of a self-similar solution of (1.6) in Q_∞ under the form

$$v_\alpha(x,t) = t^{-\frac{1+\alpha}{q-1}} V_\alpha\left(\frac{x}{\sqrt{t}}\right),$$

which satisfies

$$\lim_{t\to 0} v_\alpha(x,t) = 0 \quad \forall x \ne 0$$

and

$$\lim_{t\to 0} \int_{B_\epsilon} v_\alpha(x,t)dx = \infty \quad \forall \epsilon > 0.$$

The function V_α is nonnegative and verifies

$$-\Delta V_\alpha - \frac{1}{2}\eta.\nabla V_\alpha - \frac{1+\alpha}{q-1}V_\alpha + V_\alpha^q = 0 \quad \text{in } \mathbb{R}^N. \tag{1.10}$$