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Long-time behavior of nonlinear integro-differential evolution equations

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ABSTRACT

We study the long-time behavior as time tends to infinity of globally bounded strong solutions to certain integro-differential equations in Hilbert spaces. Based on an appropriate new Lyapunov function and the Łojasiewicz–Simon inequality, we prove that any globally bounded strong solution converges to a steady state in a real Hilbert space.

1. Introduction

In this paper we study the long-time behavior of globally bounded strong solutions to nonlinear evolutionary equations with memory of the form

$$\begin{aligned} &\frac{d}{dt}\left[k_0 u(t) + \int_0^t k_1(t-s)(u(s)-u_0)ds\right] + k_\infty u(t) + \mathcal{E}'(u(t)) = f(t), \quad t>0, \\ &u(0) = u_0, \end{aligned} \tag{1.1}$$

in a real Hilbert space H. Here $\mathcal{E}'$ is the Fréchet derivative of a functional $\mathcal{E} \in C^1(V)$, where V is a Hilbert space which densely and continuously injects into H. The vector-valued function f and the scalar kernel k_1 are locally integrable functions on $\mathbb{R}_+$; k_0, and k_∞ are real constants, and k_0 is chosen nonnegative. The vector $u_0 \in V$ stands for the initial condition for the unknown function u.

Under suitable conditions on k_1 and f, we prove that any globally bounded solution of (1.1) converges to a steady state in V, that is, $u_\infty := \lim_{t\to\infty} u(t)$ exists in V and u_∞ is a solution of the problem

$$k_\infty u_\infty + \mathcal{E}'(u_\infty) = 0,$$

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provided that the solution u has relatively compact range in V and the functional $\mathcal{E}$ satisfies the Łojasiewicz–Simon inequality (see Definition 3).

Note that, without loss of generality, we may set $k_\infty = 0$ in (1.1) since we can include it into the nonlinear term $\mathcal{E}$ by defining a new nonlinearity

$$\tilde{\mathcal{E}}(u) = \frac{k_\infty}{2}|u|_H^2 + \mathcal{E}(u).$$

In the literature one finds many papers for problems of the form (1.1), as well as variants of them which are studied in a strong setting, assuming smoothness on the nonlinearities. See e.g. [1–5]. It seems that results about convergence to steady state of globally bounded solutions of (1.1) under the general setting given above are unknown, except for some special cases, where convergence results are well-known. Examples of the latter are the following.

In case $k_0 = 1$ and $k_1 = k_\infty = 0$, Eq. (1.1) becomes a first order problem of the form

$$\begin{aligned} &u_t + \mathcal{E}'(u) = f, \quad t > 0,\\ &u(0) = u_0, \end{aligned}$$

which has been studied by many authors; see e.g. [6–8]. See also [9].

The case $k_0 = k_\infty = 0$ and $k_1 \neq 0$ of Eq. (1.1) becomes a problem of the form

$$\begin{aligned} &\frac{d}{dt}\left(\int_0^t k_1(t-s)(u-u_0)(s)ds\right) + \mathcal{E}'(u(t)) = f(t), \quad t > 0,\\ &u(0) = u_0, \end{aligned} \tag{1.2}$$

which has been studied by Vergara and Zacher [10] in a finite-dimensional Hilbert space.

In this paper we extend the results obtained in [10] to real Hilbert spaces for Eq. (1.1). More precisely, we study integro-differential evolution equations of two types.

– Problems of order 1 ($k_0 > 0$) with memory ($k_1 \neq 0$):

$$\begin{aligned} &\frac{d}{dt}\left(k_0 u(t) + \int_0^t k_1(t-s)(u(s)-u_0)ds\right) + \mathcal{E}'(u(t)) = f(t), \quad t > 0,\\ &u(0) = u_0. \end{aligned}$$

– Problems of order less than 1 ($k_0 = 0$):

$$\begin{aligned} &\frac{d}{dt}\left(\int_0^t k_1(t-s)(u(s)-u_0)ds\right) + \mathcal{E}'(u(t)) = f(t), \quad t > 0,\\ &u(0) = u_0. \end{aligned}$$

Concerning applications, let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$. Setting $H = L_2(\Omega), V = W_0^{1,2}(\Omega)$ (the classical Sobolev space) and

$$\mathcal{E}(v) = \frac{1}{2}\int_\Omega |\nabla v|^2 dx + \int_\Omega G(x, v)dx, \quad v \in W_0^{1,2}(\Omega),$$

problem (1.1) becomes a problem of the form

$$\begin{aligned} &\frac{d}{dt}\left(k_0 u(t) + \int_0^t k_1(t-s)(u(s)-u_0)ds\right) + k_\infty u(t) - \Delta u(t) + g(x, u(t)) = f(t), \quad \text{in } \mathbb{R}_+ \times \Omega,\\ &u = 0, \quad \text{on } \mathbb{R}_+ \times \partial\Omega,\\ &u(0) = u_0, \quad \text{in } \Omega, \end{aligned} \tag{1.3}$$

where $g(x, s) := \partial_s G(x, s)$ for all $s \in \mathbb{R}$, which arises as a model for nonlinear heat flow in material with memory; see e.g., Gurtin and Pipkin [11], MacCamy [12], Nunziato [13], and the monograph Prüss [4]. See also [2], where existence and uniqueness results of globally bounded solutions of (1.3) were obtained for more general functions g.

Concerning convergence to steady state for nonlinear equations with memory, there has been partial progress; see e.g. [14–20]. The reason for this lies essentially in the fact that these problems do not generate in general a semiflow in the natural phase space. Another technical difficulty consists in proving that the solutions of such problems are globally bounded and have relatively compact range in a natural phase space. Further, for problems with memory, it is in general a highly nontrivial task to construct Lyapunov functions (cf. [21, Chapter 14] and [17]) which are appropriate to investigating the asymptotic behavior of globally bounded solutions.

The paper [10] develops a method for finding Lyapunov functions for problems of order less than 1, and for orders between 1 and 2 (in time), in finite-dimensional Hilbert spaces, which combined with the Łojasiewicz inequality allows one to prove convergence to a single steady state of bounded solutions of such problems.