Nonlinear Analysis 91 (2013) 32–45

Contents lists available at SciVerse ScienceDirect

Nonlinear Analysis

journal homepage: www.elsevier.com/locate/na

The Robin eigenvalue problem for the $p(x)$-Laplacian as $p \to \infty$

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ARTICLE INFO

Article history:
Received 31 March 2013
Accepted 10 June 2013
Communicated by S. Carl

MSC:
35D30
35D40
35J20
35J60
35J66
35J70
35P30
46E30
46E35
49J40

Keywords:
∞-Laplacian
Eigenvalue problems
Luxemburg norm
$p(x)$-Laplacian
Variable exponent Lebesgue and Sobolev spaces
Viscosity solutions

ABSTRACT

We study the asymptotic behavior, as $p \to \infty$, of the first eigenvalues and the corresponding eigenfunctions for the $p(x)$-Laplacian with Robin boundary conditions in an open, bounded domain $\Omega \subset \mathbb{R}^N$ with smooth boundary. We obtain uniform bounds for the sequence of first eigenvalues (suitably rescaled), and we prove that the positive first eigenfunctions converge uniformly in Ω to a viscosity solution of a problem involving the ∞-Laplacian subject to appropriate boundary conditions.

1. Introduction

The analysis of partial differential equations with non-standard growth in the framework of variable exponent spaces has been the subject of an increasing interest during the last decade. We refer the reader to the survey by Harjulehto, Hästö, Lê & Nuortio [1] for a comprehensive account of the developments up to 2010. In particular, a lot of attention has been paid to the study of eigenvalue problems for the $p(x)$-Laplace operator

$$-\Delta_{p(x)} := -\operatorname{div}\left(|\nabla u|^{p(x)-2}\nabla u\right) = \Lambda_{p(\cdot)}|u|^{p(x)-2}u$$

in open bounded domains $\Omega \subset \mathbb{R}^N$, subject to various boundary conditions. For example, in the case of Dirichlet boundary conditions, this equation has been analyzed in [2] (see also [3]), while the Neumann and Robin boundary conditions were studied later in [4] and [5], respectively.

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0362-546X/$ – see front matter
http://dx.doi.org/10.1016/j.na.2013.06.005

More general eigenvalue problems for the $p(x)$-Laplacian have also been intensively studied in recent years. An excellent account of recent developments in this direction can be found in Mihăilescu's Ph.D. Thesis [6].

During the last several years, a number of papers have been devoted to the asymptotic analysis of solutions to partial differential equations involving the $p(x)$-Laplacian as $p(x) \to \infty$. We mention here the work of Manfredi, Rossi & Urbano [7,8], Lindqvist & Lukkari [9], Pérez-Llanos & Rossi [10,11], and Franzina & Lindqvist [12]. For the case of Dirichlet boundary conditions, the asymptotic behavior of the first eigenvalue/eigenfunction pairs associated to $-\Delta_{p(x)}$ has been studied in [10] (see also [12]), but to our knowledge the corresponding problems for other classes of boundary conditions have remained open. This paper fits into this general area of investigation. It is devoted to the study of the asymptotic behavior, as $p \to \infty$, of the first eigenvalues and the corresponding eigenfunctions for the $p(x)$-Laplacian with Robin boundary conditions:

$$\begin{cases} -\Delta_{p(x)} u = \Lambda_{p(\cdot)} |u|^{p(x)-2} u & \text{in } \Omega \\ |\nabla u|^{p(x)-2} \dfrac{\partial u}{\partial \eta} + \beta |u|^{p(x)-2} u = 0 & \text{on } \partial\Omega. \end{cases}$$

The analysis of the limiting behavior of this problem as $p \to \infty$ is undertaken here in the following sense: we replace $p = p(x)$ above by $p_n = p_n(x)$, where $\{p_n\} \subset C^1(\overline{\Omega})$ is a sequence of functions that satisfies $p_n \to \infty$, $\nabla \ln p_n \to \xi \in C(\overline{\Omega}, \mathbb{R}^N)$, and $\frac{p_n}{n} \to q \in C(\overline{\Omega}, (0, +\infty))$ uniformly in Ω, and then we study what happens with the solutions of the problems at level n as $n \to \infty$. These conditions on the sequence p_n are typical in the literature (see, e.g. [8,10,11], or [9,12] for the particular case $p_n(\cdot) = np(\cdot)$-corresponding to $\xi = \nabla \ln p$ and $q = p$). We refer the reader to Section 4 of this paper for a list of possible choices of such sequences $\{p_n\}$. We prove that after eventually extracting a subsequence, the (positive) first eigenfunctions converge uniformly in $\Omega \subset \mathbb{R}^N$ to a viscosity solution of the problem

$$\begin{cases} \min\left\{-\Delta_\infty u - |\nabla u|^2 \ln|\nabla u| \langle \xi, \nabla u\rangle, |\nabla u|^q - \Lambda_\infty |u|^q\right\} = 0 & \text{in } \Omega \\ H(x, u, \nabla u) = 0 & \text{on } \partial\Omega, \end{cases}$$

where Δ_∞ is the ∞-Laplace operator, $\Delta_\infty u := \sum_{i,j=1}^N u_{x_i} u_{x_j} u_{x_i x_j}$, Λ_∞ is the limit of the sequence of (suitably rescaled) first eigenvalues, and $H : \Omega \times [0, \infty) \times \mathbb{R}^N \to \mathbb{R}$ is given by

$$H(x, r, \theta) = \begin{cases} \max\left\{|r|^{q(x)} - |\theta|^{q(x)}, \langle \theta, \eta(x)\rangle\right\} & \text{if } r > 0 \\ \langle \theta, \eta(x)\rangle \chi_{(1,\infty)}(|\theta|) & \text{if } r = 0. \end{cases}$$

The plan of the paper is as follows. In Section 2 we review the definition and some basic properties of variable exponent Lebesgue and Sobolev spaces that will be needed in the sequel. Section 3 of the paper is devoted to the Robin eigenvalue problem for $-\Delta_{p(x)}$ for the case where $p = p(x)$ is fixed. After recalling the definition of a weak solution, we revisit some details concerning the Ljusternik–Schnirelman existence theory for this problem, and we show that continuous weak solutions are also solutions in the viscosity sense. Here, we adopt the definition of viscosity solutions for second-order elliptic equations with fully nonlinear boundary conditions introduced by Barles in [13]. Finally, in Section 4 we state and prove the main result of the paper, Theorem 1, regarding the convergence of the first eigenvalues and the corresponding positive eigenfunctions as $p(\cdot) \to \infty$.

2. Preliminaries

In this section, we provide a brief introduction to variable exponent Lebesgue and Sobolev spaces. For more details we refer the reader to the books by Diening, Harjulehto, Hästö & M. Růžička [14], Musielak [15], and the papers by Edmunds, Lang & Nekvinda [16], Edmunds & Rákosník [17,18], and Kovacik & Rákosník [19].

Let $\Omega \subset R^N$ be an open set with smooth boundary, and let $|\Omega|$ stand for the N-dimensional Lebesgue measure of Ω. Given any continuous function $p : \overline{\Omega} \to (1, \infty)$, let $p^- := \inf_{x\in\Omega} p(x)$ and $p^+ := \sup_{x\in\Omega} p(x)$. The variable exponent Lebesgue space $L^{p(\cdot)}(\Omega)$ is defined by

$$L^{p(\cdot)}(\Omega) = \left\{u : \Omega \to \mathbb{R} \text{ measurable} : \int_\Omega |u(x)|^{p(x)}\, dx < \infty\right\}.$$

It is a Banach space when endowed with the so-called Luxemburg norm

$$|u|_{p(\cdot)} := \inf\left\{\mu > 0 : \int_\Omega \left|\frac{u(x)}{\mu}\right|^{p(x)} dx \le 1\right\}.$$

For constant functions p the space $L^{p(\cdot)}(\Omega)$ reduces to the classical Lebesgue space $L^p(\Omega)$, endowed with the standard norm

$$\|u\|_{L^p(\Omega)} := \left(\int_\Omega |u(x)|^p dx\right)^{1/p}.$$

$L^{p(\cdot)}(\Omega)$ is separable and reflexive if $1 < p^- \le p^+ < +\infty$. If $0 < |\Omega| < \infty$ and if p_1, p_2 are variable exponents such that $p_1 \le p_2$ in Ω then the embedding $L^{p_2(\cdot)}(\Omega) \hookrightarrow L^{p_1(\cdot)}(\Omega)$ is continuous, and its norm does not exceed $|\Omega| + 1$.