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On solutions of quasilinear elliptic inequalities containing terms with lower-order derivatives ☆

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ABSTRACT

We obtain estimates and blow-up conditions for solutions of quasilinear elliptic inequalities containing terms with lower-order derivatives.

1. Introduction

Let Ω be an unbounded open subset of $\mathbb{R}^n$, $n \geq 2$. As in [1], by $W^1_{p,\mathrm{loc}}(\Omega)$, $p > 1$, we mean the space of measurable functions that belong to $W^1_p(B_r \cap \Omega)$ for all real numbers $r > 0$ satisfying the condition $B_r \cap \Omega \neq \emptyset$, where B_r is the open ball in $\mathbb{R}^n$ of radius r and center at zero. The space $L_{\infty,\mathrm{loc}}(\Omega)$ is defined in a similar way.

Consider the inequality

$$\operatorname{div} A(x, Du) + b(x)|Du|^{p-1} \geq q(x)g(u) \quad \text{in } \Omega, \tag{1.1}$$

where $D = (\partial/\partial x_1, \ldots, \partial/\partial x_n)$ is the gradient operator, $b \in L_{\infty,\mathrm{loc}}(\Omega)$, and $A : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ is a measurable function such that

$$C_1|\zeta|^p \leq \zeta A(x, \zeta), \qquad |A(x, \zeta)| \leq C_2|\zeta|^{p-1}$$

with some constants $C_1 > 0$, $C_2 > 0$, and $p > 1$ for almost all $x \in \Omega$ and for all $\zeta \in \mathbb{R}^n$.

We assume that $S_r \cap \Omega \neq \emptyset$ for all $r > r_0$, where $r_0 > 0$ is some real number and S_r is the sphere in $\mathbb{R}^n$ of radius r and center at zero. Also let $q \in L_{\infty,\mathrm{loc}}(\Omega)$ and $g \in C([0, \infty))$ be non-negative functions and, moreover, $g(t) > 0$ for all $t > 0$. We denote

$$f_\sigma(r) = \frac{\operatorname*{ess\,inf}_{\Omega_{r/\sigma,\sigma r}} q}{1 + r \operatorname*{ess\,sup}_{\Omega_{r/\sigma,\sigma r}} |b|}, \quad r > r_0, \ \sigma > 1,$$

$$q_\sigma(r) = \operatorname*{ess\,inf}_{\Omega_{r/\sigma,\sigma r}} q, \quad r > r_0, \ \sigma > 1$$

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and

$$g_\theta(t) = \inf_{(t/\theta, \theta t)} g, \quad t > 0,\ \theta > 1,$$

where $\Omega_{r_1,r_2} = \{x \in \Omega : r_1 < |x| < r_2\}$, $0 \le r_1 < r_2 \le \infty$.

A non-negative function $u \in W^1_{p,\mathrm{loc}}(\Omega) \cap L_{\infty,\mathrm{loc}}(\Omega)$ is called a solution of inequality (1.1) if the map $x \mapsto A(x, Du)$ is measurable and

$$-\int_\Omega A(x, Du) D\varphi \, dx + \int_\Omega b(x)|Du|^{p-1}\varphi \, dx \ge \int_\Omega q(x) g(u) \varphi \, dx$$

for any non-negative function $\varphi \in C_0^\infty(\Omega)$. As is customary, the condition

$$u|_{\partial\Omega} = 0 \tag{1.2}$$

means that $\psi u \in \mathring{W}^1_p(\Omega)$ for any $\psi \in C_0^\infty(\mathbb{R}^n)$. If $\Omega = \mathbb{R}^n$, then (1.2) is obviously valid for all $u \in W^1_{p,\mathrm{loc}}(\mathbb{R}^n)$.

For every solution of (1.1), (1.2) we put

$$M(r; u) = \operatorname*{ess\,sup}_{S_r \cap \Omega} u, \quad r > r_0,$$

where the restriction of u to $S_r \cap \Omega$ is understood in the sense of the trace and the ess sup on the right-hand side is with respect to $(n-1)$-dimensional Lebesgue measure on S_r.

Starting from the classical papers of J.B. Keller [2] and R. Osserman [3], the problem of nonexistence of nontrivial solutions for differential equations and inequalities, which is known as the blow-up problem, captures the attention of many mathematicians. Our research deals with a priori estimates and blow-up conditions for solutions of (1.1), (1.2). In the case that $p = 2$ and $A(x, \cdot)$ is a linear operator with the matrix $\|a_{ij}(x)\|$, where a_{ij}, $i, j = 1, \dots, n$, are bounded measurable functions, inequality (1.1) takes the form

$$\sum_{i,j=1}^n \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right) + b(x)|Du| \ge q(x) g(u) \quad \text{in } \Omega. \tag{1.3}$$

It can easily be seen that (1.3) is valid for any solution of the inequality

$$\sum_{i,j=1}^n \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right) + \sum_{i=1}^n b_i(x)\frac{\partial u}{\partial x_i} \ge q(x) g(u) \quad \text{in } \Omega,$$

where b_i, $i = 1, \dots, n$, are locally bounded measurable functions, if we put

$$b(x) = \left(\sum_{i=1}^n |b_i(x)|^2\right)^{1/2}.$$

The questions treated in our article were investigated mainly for $g(t) = t^\lambda$ [4–10]. The case of general nonlinearity without lower-order derivatives was studied in [11,12,2,13,3]. In [14], blow-up conditions were obtained for solutions of inequalities with general nonlinearity and lower-order derivatives. However, results of [14] cannot be applied to inequalities considered in Examples 2.1–2.3.

We also note that the authors of [11,14,12,2,3] use arguments based on the method of barrier functions. This method is unusable for inequalities of the general form (1.1). It is unusable even for inequalities with measurable coefficients of the form (1.3). Theorems 2.1–2.4 proved below do not have this drawback.

2. Main results

Theorem 2.1. *Let*

$$\int_1^\infty (g_\theta(t) t)^{-1/p} \, dt < \infty \tag{2.1}$$

and

$$\int_{r_0}^\infty (r f_\sigma(r))^{1/(p-1)} \, dr = \infty \tag{2.2}$$

for some real numbers $\theta > 1$ and $\sigma > 1$, then any solution of (1.1), (1.2) is trivial, i.e. $u = 0$ almost everywhere in Ω.

Remark 2.1. We remind that, by definition, all solutions of (1.1), (1.2) are non-negative functions since the domain of the function g is the interval $[0, \infty)$.