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On sets of occupational measures generated by a deterministic control system on an infinite time horizon

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ABSTRACT

We give a representation for the closed convex hull of the set of discounted occupational measures generated by control-state trajectories of a deterministic control system. We also investigate the limit behavior of the latter when the discount factor tends to zero and compare it with the limit behavior of the long run time average occupational measures set. The novelty of our results is in that we allow the control set dependence on the state variables that make the results to be applicable to differential inclusions.

1. Introduction and preliminaries

It is well known that nonlinear optimal control problems can be equivalently reformulated as infinite dimensional linear programming problems considered on spaces of occupational measures generated by control-state trajectories. Having many attractive features and being applicable in both stochastic and deterministic settings, the linear programming (**LP**) based approaches to optimal control problems have been intensively studied in the literature. Important results justifying the use of LP formulations in dealing with various problems of optimal control of stochastic systems were obtained in [1–5] (see also more recent developments in [6–9]). Various aspects of LP based analysis and solution of deterministic optimal control problems on a finite time horizon were studied in [10–15] (and in some earlier works mentioned therein). The LP approach to deterministic optimal control problems with long run time average and time discounting criteria was developed in [16–20].

This paper continues the line of research started in [16–18], with the main focus (and novelty) of the results presented here being that the control set is allowed to be dependent on the state variables (which extends the applicability of the LP approach to differential inclusions).

We consider the control system

$$y'(t) = f(u(t), y(t)), \quad t > 0, \qquad y(0) = y_0, \tag{1}$$

where the controls are measurable functions satisfying the inclusion

$$u(t) \in U(y(t)). \tag{2}$$

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Here $f(u, y) : M \times \mathbb{R}^m \mapsto \mathbb{R}^m$ is a continuous function of (u, y) that satisfies local Lipschitz conditions in y uniformly with respect to $u \in M$, M is a compact metric space and $U(\cdot)$ is an upper semicontinuous set-valued function that maps points in $\mathbb{R}^m$ into closed subsets of M ($U(y) \subset M$). Solutions of the system (1) will be assumed to satisfy the inclusion

$$y(t) \in Y \quad \forall t \in [0, \infty), \tag{3}$$

where Y is a given compact subset of $\mathbb{R}^m$, with (3) being interpreted as a state constraint.

The paper presents results about sets of occupational measures generated by the solutions of the system (1)–(2). Firstly, we obtain a representation formula for the closed convex hull of the set of discounted occupational measures generated by the control-state trajectories of the system (1)–(2). This result is stated in Section 2.1 (see Theorem 2.2) and proved in Section 3 (the idea of the proof being similar to that of [15], where optimal control problems on a finite time horizon were considered). Secondly, we use the above mentioned result to obtain a representation formula for the limit of the closed convex hull of the set of "non-discounted" occupational measures (these being simply called occupational measures) generated by the control-state trajectories of the system (1)–(2) on a finite time interval $[0, S]$ when $S \to \infty$. These results are stated in Section 2.2 (see Theorems 2.5, 2.6 and Proposition 2.7) and proved in Section 4.

Let us introduce some notations and definitions that are going to be used in the paper. Given a compact metric space X, the space of probability measures defined on Borel subsets of X will be denoted as $\mathcal{P}(X)$. Note that, being endowed with a metric consistent with its weak* topology, $\mathcal{P}(X)$ becomes a compact metric space. Given such a metric ρ, one can define the "distance" $\rho(\gamma, \Gamma)$ between $\gamma \in \mathcal{P}(X)$ and $\Gamma \subset \mathcal{P}(X)$ and the Hausdorff metric $\rho_H(\Gamma_1, \Gamma_2)$ between $\Gamma_1 \subset \mathcal{P}(X)$ and $\Gamma_2 \subset \mathcal{P}(X)$ as follows:

$$\rho(\gamma, \Gamma) \stackrel{\text{def}}{=} \inf_{\gamma' \in \Gamma} \rho(\gamma, \gamma'), \qquad \rho_H(\Gamma_1, \Gamma_2) \stackrel{\text{def}}{=} \max\{\sup_{\gamma \in \Gamma_1} \rho(\gamma, \Gamma_2), \sup_{\gamma \in \Gamma_2} \rho(\gamma, \Gamma_1)\}. \tag{4}$$

A pair $(u(t), y(t))$ will be called y_0-*admissible* on a finite time interval $[0, S]$ ($S > 0$), or on the infinite time interval $[0, \infty)$ if (1)–(2) are satisfied for almost all $t \in [0, S]$ and $y(t) \in Y$ for all $t \in [0, S]$ (respectively, for almost all and all $t \in [0, \infty)$).

Let

$$K \stackrel{\text{def}}{=} Graph(U) = \{(u, y) : u \in U(y), y \in Y\}. \tag{5}$$

Note that, due to the upper semicontinuity of $U(\cdot)$, K is a compact subset of $M \times Y$. Given an y_0-admissible pair $(u(\cdot), y(\cdot))$, the probability measure $\gamma \in \mathcal{P}(K)$ is called the occupational measure generated by this pair on an interval $[0, S]$ if

$$\int_K h(u, y)\gamma(du, dy) = \frac{1}{S}\int_0^S h(u(t), y(t))dt \tag{6}$$

for any $h(\cdot) \in C(K)$ (the space of continuous functions on K). The probability measure $\gamma \in \mathcal{P}(K)$ is called the discounted occupational measure generated by this pair if

$$\int_K h(u, y)\gamma(du, dy) = C\int_0^\infty e^{-Ct} h(u(t), y(t))dt \tag{7}$$

for any $h(\cdot) \in C(K)$.

Along with the system (1)–(2), let us consider a relaxed version of this system

$$y'(t) = \bar{f}(\mu(t), y(t)), \quad t > 0, \qquad y(0) = y_0, \tag{8}$$

where the controls are measurable functions $\mu(t) = \mu(t, du) \in \mathcal{P}(U(y(t)))$ and

$$\bar{f}(\mu(t), y(t)) \stackrel{\text{def}}{=} \int_{U(y(t))} f(u, y(t))\mu(t, du). \tag{9}$$

A pair $(\mu(t), y(t))$ will be called *relaxed* y_0-*admissible* on a finite time interval $[0, S]$ ($S > 0$), or on the infinite time interval $[0, \infty)$ if (8)–(9) are satisfied for almost all $t \in [0, S]$ and $y(t) \in Y$ for all $t \in [0, S]$ (respectively, for almost all and all $t \in [0, \infty)$).

Given a relaxed y_0-admissible pair $(\mu(\cdot), y(\cdot))$, the probability measure $\gamma \in \mathcal{P}(K)$ is called the occupational measure generated by this pair on an interval $[0, S]$ if

$$\int_K h(u, y)\gamma(du, dy) = \frac{1}{S}\int_0^S \left(\int_{U(y(t))} h(u, y(t))\mu(t, du)\right) dt \tag{10}$$

for any $h(\cdot) \in C(K)$. The probability measure $\gamma \in \mathcal{P}(K)$ is called the discounted occupational measure generated by this pair if

$$\int_K h(u, y)\gamma(dy, du) = C\int_0^\infty e^{-Ct} \left(\int_{U(y(t))} h(u, y(t))\mu(t, du)\right) dt, \tag{11}$$

for any $h(\cdot) \in C(K)$.