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A generalization of the Caginalp phase-field system with Neumann boundary conditions

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ABSTRACT

We study a generalized Caginalp phase-field system based on the theory of type III heat conduction proposed by Green and Naghdi and supplemented with Neumann boundary conditions. In contrast to the Dirichlet case, the system exhibits a lack of dissipation on the thermal displacement variable α. However, α minus its spatial average is dissipative and we are able to prove the existence of the global attractor with optimal regularity for the associated semigroup.

1. Introduction

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with smooth boundary $\partial\Omega$. We consider the following phase-field system of Caginalp type in the unknowns $u = u(\boldsymbol{x}, t) : \Omega \times \mathbb{R}^+ \to \mathbb{R}$ and $\alpha = \alpha(\boldsymbol{x}, t) : \Omega \times \mathbb{R}^+ \to \mathbb{R}$:

$$\begin{cases} u_t - \Delta u + \phi(u) = \alpha_t, \\ \alpha_{tt} - \Delta\alpha_t - \Delta\alpha = -u_t, \end{cases} \tag{1.1}$$

supplemented with the initial data

$$\begin{cases} u(\boldsymbol{x}, 0) = u_0(\boldsymbol{x}), \\ \alpha(\boldsymbol{x}, 0) = \alpha_0(\boldsymbol{x}), \\ \alpha_t(\boldsymbol{x}, 0) = \alpha_1(\boldsymbol{x}) \end{cases}$$

and homogeneous Neumann boundary conditions

$$\begin{cases} \partial_{\mathbf{n}} u(\boldsymbol{x}, t) = 0, \quad \boldsymbol{x} \in \partial\Omega, \\ \partial_{\mathbf{n}} \alpha(\boldsymbol{x}, t) = 0, \quad \boldsymbol{x} \in \partial\Omega. \end{cases} \tag{1.2}$$

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System (1.1) arises as a model for solid–liquid phase transition within the theory of heat conduction proposed by Green and Naghdi [1–5]. In this context, u is the order parameter and satisfies the usual parabolic evolution equation

$$u_t - \Delta u + \phi(u) = \theta,$$

where θ is the relative temperature of the material and ϕ is the derivative of a double-well potential Φ. Furthermore, α is a new independent variable, called *thermal displacement*, defined as

$$\alpha(t) := \int_0^t \theta(\tau)\,\mathrm{d}\tau + \alpha(0),$$

where $\alpha(0)$ is assumed to be known. Then, the heat conduction law of type III prescribes a constitutive equation for the heat flux $\boldsymbol{q}$ of the form

$$\boldsymbol{q} = -k^\star \nabla\alpha - k\nabla\theta, \quad k, k^\star > 0,$$

instead of the classical Fourier law

$$\boldsymbol{q} = -k\nabla\theta, \quad k > 0.$$

As a result, the evolution law

$$\frac{\partial H}{\partial t} + \nabla \cdot \boldsymbol{q} = 0$$

for the enthalpy

$$H := \theta + u = \alpha_t + u$$

becomes the second-order (in time) equation for α in (1.1).

One motivation for considering the type III law over the Fourier law is that the latter predicts that thermal signals propagate with an infinite speed, which violates causality (see, e.g., [6]). Other alternatives have been proposed and studied in [7–12]; one essential feature of these alternative models is that one ends up with a second-order (in time) equation for the temperature/the thermal displacement variable.

We also refer the reader to [13–24] for discussions and other developments related to type III thermoelasticity.

System (1.1) has been studied in [25,26] in the case of Dirichlet boundary conditions. In particular, the well-posedness and the existence of regular global attractors have been proved.

In this paper, we consider the physically relevant Neumann boundary conditions both for u and the temperature, hence, equivalently, for α. A first difference, compared to the Dirichlet case, is that the spatial average of the enthalpy H in the bulk is preserved by the evolution. Indeed, integrating the second equation in (1.1) over Ω and taking into account the homogeneous Neumann boundary condition on α, one formally obtains

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega H(\boldsymbol{x}, t)\,\mathrm{d}\boldsymbol{x} = 0,$$

hence the conservation law

$$\langle H(t)\rangle = \langle \alpha_1 + u_0\rangle, \quad \forall t \geq 0, \tag{1.3}$$

where $\langle H\rangle = \frac{1}{|\Omega|}\int_\Omega H\,\mathrm{d}\boldsymbol{x}$ will also be called enthalpy. On the other hand, compared again to the Dirichlet case, the system is *not* dissipative in general. Indeed, if $\phi(u^\star) = c^\star$ for some $c^\star \neq 0$ and $u^\star \in \mathbb{R}$, then the pair $(u^\star, \alpha^\star)$ with $\alpha^\star = c^\star t + \alpha_0$ is a solution to (1.1), but $\alpha^\star$ is unbounded as $t \to +\infty$, against the definition of dissipativity. This phenomenon has already been observed in [12] for a phase-field system based on the Maxwell–Cattaneo law and suggests to focus instead on the evolution of

$$\bar\alpha = \alpha - \langle\alpha\rangle.$$

The main result of this paper states that the continuous operator associated with (1.1) by the rule

$$(u_0, \bar\alpha_0, \alpha_1) \mapsto (u(t), \bar\alpha(t), \alpha_t(t))$$

is a dissipative semigroup $S(t)$ on a suitable phase space and possesses the global attractor $\mathbb{A}$ (note that, in the case of the Maxwell–Cattaneo law, we do not have a semigroup in general; see [12]). Furthermore, we prove that this attractor is of optimal regularity.

We recall that the global attractor $\mathbb{A}$ is the smallest (for the inclusion) compact set of the phase space which is invariant by the flow (i.e., $S(t)\mathbb{A} = \mathbb{A},\ \forall t \geq 0$) and attracts all bounded sets of initial data as time goes to infinity; it thus appears as a suitable object in view of the study of the asymptotic behavior of the system (see, e.g., [27–29] for thorough discussions and applications).

Plan of the paper. We provide the suitable functional setting and the assumptions on ϕ in Section 1. Then, Section 2 is devoted to the proof of the basic energy estimates which lead in the following Section 3 to the well-posedness of the problem. Besides, we construct a strongly continuous semigroup $S(t)$ by restricting our attention to thermal displacement variables with zero average. In Section 4–Section 5, we study the dissipativity of this semigroup, showing a partial regularization effect both on u and α_t. Finally, in Section 6, we discuss the existence and regularity of the global attractor for $S(t)$.