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Profile decompositions and blowup phenomena of mass critical fractional Schrödinger equations

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ABSTRACT

We study, under the radial symmetry assumption, the solutions to the fractional Schrödinger equations of critical nonlinearity in $\mathbb{R}^{1+d}$, $d \geq 2$, with Lévy index $2d/(2d-1) < \alpha < 2$. We first prove the linear profile decomposition and then apply it to investigate the properties of the blowup solutions of the nonlinear equations with mass-critical Hartree type nonlinearity.

1. Introduction

In [1], Laskin introduced the fractional quantum mechanics in which he generalized the Brownian-like quantum mechanical path, in the Feynman path integral approach to quantum mechanics, to the α-stable Lévy-like quantum mechanical path. This gives rise to the fractional generalization of the Schrödinger equation. Namely, the associated equation for the wave function results in the fractional Schrödingier equations, which contain a nonlocal fractional derivative operator $(-\Delta)^{\frac{\alpha}{2}}$ defined by $(-\Delta)^{\frac{\alpha}{2}} = \mathcal{F}^{-1}|\xi|^\alpha \mathcal{F}$. In this paper we consider the following Cauchy problem with mass critical Hartree type nonlinearity:

$$\begin{cases} iu_t + (-\Delta)^{\frac{\alpha}{2}} u = \lambda(|x|^{-\alpha} * |u|^2)u, & (t, x) \in \mathbb{R}^{1+d},\ d \geq 2, \\ u(0, x) = f(x) \in L^2, \end{cases} \tag{1.1}$$

where $\lambda = \pm 1$. Here α is the Lévy stability index with $1 < \alpha \leq 2$. When $\alpha = 2$, the fractional Schrödinger equation becomes the well-known Schrödinger equation. See [2,3] and references therein for further discussions related to the fractional quantum mechanics.

The solutions to Eq. (1.1) have the conservation laws for the mass and the energy:

$$M(u) = \int |u|^2\, dx, \qquad E(u) = \frac{1}{2}\int \overline{u}|\nabla|^\alpha u\, dx - \frac{\lambda}{4}\int \overline{u}(|x|^{-\alpha} * |u|^2)u\, dx.$$

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We say that (1.1) is focusing if $\lambda = 1$, and defocusing if $\lambda = -1$. The Eq. (1.1) is mass-critical, as $M(u)$ is invariant under scaling symmetry $u_\rho(t,x) = \rho^{-d/2}u(t/\rho^\alpha, x/\rho)$, $\rho > 0$ which is again a solution to (1.1) with initial datum $\rho^{-d/2}u(0, x/\rho)$. The Eq. (1.1) is locally well-posed in L^2 for radial initial data and globally well-posed for sufficiently small radial data [4]. (See [5] for results regarding power type nonlinearities.) For the focusing case, the authors [6] used a virial argument to show the finite time blowup, with radial data, provided that the energy $E(u)$ is negative. Also see [7] and [8] for results with noncritical nonlinearity.

In this paper we aim to investigate the blowup phenomena of (1.1) with radial data when $\alpha < 2$. Due to the critical nonlinearity the time of existence no longer depends on the L^2 norm of initial data. Instead it relies on the profiles of the data. Hence the situation become more subtle. When $\alpha = 2$, a lot of work was devoted to the study of blowup phenomena, which was based on the usual Strichartz and its refinements. (See for instance [9–11].) When it comes to the fractional the Schrödinger equation ($1 < \alpha < 2$), due to the nonlocality of fractional operator, various useful properties (e.g. Galilean invariance) which hold for the Schrödinger equation are no longer available. The main difficulty comes from absence of proper linear estimates. In fact, by scaling the condition $\alpha/q + d/r = d/2$ should be satisfied by the pair (q, r) if L^2-$L_t^qL_x^r$ estimates were true for the linear propagator $f \to e^{it(-\Delta)^{\frac{\alpha}{2}}}f$. But such estimate is impossible as the Knapp example shows that $\dot{H}^s$-$L_t^qL_x^r$ is only possible for $2/q + d/r \le d/2$. In order to get around these difficulties we work with radial assumption on the initial data, which allows us to use the recent results on the Strichartz estimates for radial functions [5] or angularly regular functions [12].

Linear profile decomposition

As for linear estimates such as the Strichartz estimates or Sobolev inequalities, the presence of noncompact symmetries causes defect of compactness. The profile decomposition with respect to the associated linear estimates is a measure to make it rigorous that such symmetries are the only source of noncompactness.

Concerning nonlinear dispersive equations (especially nonlinear wave and Schrödinger equations), the profile decompositions have been intensively studied and led to various recent developments in the study of equations with the critical nonlinearity ([9]). Profile decompositions for the Schrödinger equations with L^2 data were obtained by Merle and Vega [11] when $d = 2$, Carles and Keraani [13], $d = 1$, and Bégout and Vargas [14], $d \ge 3$. (Also see [15–17] for results on the wave equation and [18–20] on general dispersive equations.) These results are based on refinements of Strichartz estimates (see [21,14]). There is a different approach based on Sobolev embedding but such approach is not applicable especially the equation is L^2-critical. Our approach is also based on a refinements of Strichartz estimate. Thanks to the extended range of admissible due to the radial assumption it is relatively simpler to obtain the refinement (see Proposition 2.3 which is used for the proof of profile decomposition.)

We now define the linear propagator $U(t)f$ to be the solution to the linear equation $iu_t + (-\Delta)^{\frac{\alpha}{2}}u = 0$ with initial datum f. Then it is formally given by

$$U(t)f = e^{it(-\Delta)^{\frac{\alpha}{2}}}f = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{i(x\cdot\xi + t|\xi|^\alpha)}\widehat{f}(\xi)\,d\xi. \tag{1.2}$$

Here $\widehat{f}$ denotes the Fourier transform of f such that $\widehat{f}(\xi) = \int_{\mathbb{R}^d} e^{-ix\cdot\xi}f(x)\,dx$.

The following is our first result:

Theorem 1.1. *Let $d \ge 2$, $\frac{2d}{2d-1} < \alpha < 2$, and $2 < q, r < \infty$ satisfy $\frac{\alpha}{q} + \frac{d}{r} = \frac{d}{2}$. Suppose that $(u_n)_{n\ge1}$ is a sequence of complex-valued radial functions satisfying $\|u_n\|_{L_x^2} \le 1$. Then up to a subsequence, for any $l \ge 1$, there exist a sequence of radial functions $(\phi^j)_{1\le j\le l} \in L^2$, $\omega_n^l \in L^2$ and a family of parameters $(h_n^j, t_n^j)_{1\le j\le l, n\ge1}$ such that*

$$u_n(x) = \sum_{1\le j\le l} U(t_n^j)\left[(h_n^j)^{-d/2}\phi^j(\cdot/h_n^j)\right](x) + \omega_n^l(x)$$

and the following properties are satisfied:

$$\lim_{l\to\infty}\limsup_{n\to\infty}\|U_\alpha(\cdot)\omega_n^l\|_{L_t^qL_x^r(\mathbb{R}\times\mathbb{R}^d)} = 0,$$

and for $j \ne k$, $(h_n^j, t_n^j)_{n\ge1}$ and $(h_n^k, t_n^k)_{n\ge1}$ are asymptotically orthogonal in the sense that

$$\text{either } \limsup_{n\to\infty}\left(\frac{h_n^j}{h_n^k} + \frac{h_n^k}{h_n^j}\right) = \infty,$$

$$\text{or } (h_n^j) = (h_n^k) \quad \text{and} \quad \limsup_{n\to\infty}\frac{|t_n^j - t_n^k|}{(h_n^j)^\alpha} = \infty,$$

and for each l

$$\lim_{n\to\infty}\left[\|u_n\|_{L_x^2}^2 - \left(\sum_{1\le j\le l}\|\phi^j\|_{L_x^2}^2 + \|\omega_n^l\|_{L^2}^2\right)\right] = 0.$$